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Study Material

On Ordinary Differential Equation of Higher Order which includes:
Linear differential equation of nth order with constant coefficients, Simultaneous linear differential equations, Second order linear differential equations with variable coefficients, Solution by changing independent variable, Reduction of order, Normal form, Method of variation of parameters, Cauchy-Euler equation, Series solutions (Frobenius Method).

Differential Equation:

Consider the area of a rectangle A = x× y so area depends upon the length and breadth of rectangle. So changing the length and breadth of the rectangle we get a new area or area changed. So Area of a rectangle depends on length and breadth of rectangle. So here length and breadth are independent variable and Area is dependent variable.

$$A = f(x, y)$$

Similarly, we can take y = f(x).

The differentiation of y with respect to x is called derivative of y with respect to x.

An equation involving independent variable, dependent variable and their derivative with respect to independent variable is called a differential equation.

For example, you may consider.

(1) $\frac{dy}{dx}=\frac{1+x^2}{1+y^2}$

(2) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+y=0$

(3) $\frac{d^2y}{dx^2}+\sqrt{1+\left(\frac{dy}{dx}\right)^3}=0.$

are differential equations, since they contains x as in independent variable, y as dependent variable and $\frac{dy}{dx}$ the derivative of y with respect to x.

<u>Ordinary Differential Equation</u>: If the differential equation contains only one independent variable then it is called an ordinary differential equation. If it contains more than one independent variable then it is called partial differential equation (1), (2) are ordinary differential equation and for partial differential equation you may consider $y\frac{\partial A}{\partial x}+x\frac{\partial A}{\partial y}=C^x$ is partial differential equation as dependent variable A contains two independent variable x and y.

<u>Order and Degree of a Differential Equation</u>: Order of a differential equation is the order of highest derivative involved in the equation. For example in (2) the derivatives are $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and in the these two highest order derivative is $\frac{d^2y}{dx^2}$ so order of this differential equation is 2.

Order of (1) is 1.

<u>Degree of a Differential Equation</u>: Degree of a differential Equation is degree of highest order derivative involved in the equation when equation is free from radical and fractional powers.

Example. Differential equation (1) & (2) are of first degree and (3) is of second degree as $\frac{d^2y}{dx^2}+\sqrt{1+\left(\frac{dy}{dx}\right)^3}=0 \Rightarrow \left(\frac{dy}{dx}\right)^2+1+\left(\frac{dy}{dx}\right)^3=0$.

<u>Linear Differential equation</u> : A differential of the form $\frac{dy}{dx}+Py=Q$

where P and Q are functions of independent variable x (only) (But not y or constants) is called a linear Differential Equation.

It is said to be linear because the dependent variable y and its derivative w.r.t. x occurs only in Ist degree.

Working Rule to solve linear differential equation of Ist order:

1. Arrange Given differential equation in the form $\frac{dy}{dx}+py=Q.$
2. Write down its I.F. $=e^{\int P\,dx}$.

3. General solution of differential equation is $y.(I.F.)=\int Q(I.F.)\,dx+c.$

Example.: Solve the differential equation $\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^4}.$

Solution : Given differential equation is in Linear form with $P=\frac{3}{x}$ and $Q=\frac{1}{x^4}.$

Hence I.F. $=e^{\int P\,dx}=e^{\int Pdx}=e^{3\int\frac{1}{x}dx}=e^{3\log x}=x^3$

Therefore General solution of differential equation is $y.(I.F.)=\int Q(I.F.)\,dx+c.$

$y.(I.F.)=\int Q(I.F.)\,dx+c \Rightarrow yx^3=\int\frac{1}{x^4}(x^3)dx + c$ where c is arbitrary constant of integration.

Example: Solve the differential equation $(1+x^2)\frac{dy}{dx}+2xy=\text{Cos}\,x$.

Solution : Given differential equation can be written as $\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\text{Cos}\,x}{1+x^2}$

which is a linear differential equation with $P=\frac{2x}{1+x^2}$ & $Q=\frac{\text{Cos}\,x}{1+x^2}$

and its integrating factor is $e^{\int\frac{2x}{1+x^2}}dx=1+x^2$.

Hence Required solution is

$$y\left(1+x^2\right)=c+\int\frac{\text{Cos}\,x}{1+x^2}\left(1+x^2\right)dx=c+\int\text{Cos}\,x=C+\text{Sin}\,x$$

or $y=\frac{c+\text{Sin}\,x}{1+x^2}.$

Equation Reducible to Linear form (Bernoulli's form)

Bernoulli equation is the form $\frac{dy}{dx}+py=Qy^n$ ....(1)

where P and Q are functions of x only.

This can be reduced to linear form by dividing it by y^n and substituting

$$\frac{1}{y^{n-1}}=v \;\; or \;\; y^{1-n}=v$$

Dividing by y^n equation (1) becomes $y^{-n}\frac{dy}{dx}+py^{-n+1}=Q$ ....(2)

Now put $v = y^{1-n} \Rightarrow \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

so (2) becomes $\frac{1}{(1-n)}\frac{dv}{dx} + pv = Q$

or $\frac{dv}{dx} + p(1-n)v = Q(1-n)$

which is a linear equation and can be solved.

Problem : Solve the differential equation $x\frac{dy}{dx} + y = xy^3$

Solution : Given equation can be written as $y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = x$ .... (1)

Now put $y^{-2} = v \Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

So (1) becomes $-\frac{1}{2}\frac{dv}{dx} + \frac{v}{x} = 1$

or $\frac{dv}{dx} - 2\frac{v}{x} = -2$ .... (2)

which is a linear equation

and its I.F. $= e^{\int -2/x\, dx} = e^{-2\log x} = e^{\log\frac{1}{x^2}} = \frac{1}{x^2}$

so solution of equation (2) is $v.\frac{1}{x^2} = \int -2/x\, dx + c$

putting the value of v we have $\frac{1}{x^2y^2} = \frac{2}{x} + c$ which is required solution.

<u>Exact Differential Equation</u>: If a differential equation is obtained by direct differentiation of its primitive (solution) without any other process like elimination or reduction then it is exact differential equation. For example, x dy + y dx = 0 is an exact differential equation since it is obtained by direct differentiation of xy = c^2 which is its primitive.

In other words, (W.M.) : A differential equation Mdx + N dy = 0 is said to be an exact differential equation if the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is satisfied, where $\frac{\partial M}{\partial y}$ is differential coefficient of M w.r.t y keeping x constant

and $\frac{\partial N}{\partial x}$ is differential coefficient of M w.r.t x keeping y constant.

Method for solving exact Differential equation (W.M.):

Step-I: Integrate M w.r.t. x keeping y as constant.

Step-II: Integrate w.r.t. y, those terms of N which not contain x.

Step-III: Result I + Result II = Constant is the solution.

Example: Solve the differential equation

$\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0$.

Solution : Here $M=5x^4+3x^2y^2-2xy^3$, & $N=2x^3y-3x^2y^2-5y^4$

Check

so $\frac{\partial M}{\partial y}=6x^2y-6xy^2$ & $\frac{\partial N}{\partial x}=6x^2y-6xy^2$

since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ hence given equation is exact.

Now $\int M\,(treating\ y\ as\ const\,\mathrm{an}\,t)\ dx+\int\left(\text{items of N not conting x}\right)dy=C$

or $\int\left(5x^4+3x^2y^2-2xy^3\right)dx+\int\left(-5y^4dy\right)=C$

$\int\left(5x^4+3x^2y^2-2xy^3\right)dx+\int\left(-5y^4dy\right)=C$

$5\frac{x^5}{5}+3y^2\frac{x^3}{3}-2y^3\frac{x^2}{2}-5y^4\frac{y^5}{5}=C$

$x^5+x^3y^2-x^2y^3-y^5=C$.

<u>Integrating factor</u>: Sometimes an equation which is not exact can be made exact differential equation, by multiplying some suitable function of x and y. This function is known as integrating factor of the differential equation.

For example $(y-x)x^2\,\mathrm{Sin}\,ydy+\left(1+x^2\right)dx=0$ is not exact as $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

But after multiplying by $\frac{1}{x^2}$ given equation becomes exact differential equation.

<u>Rules for Finding Integrating Factor</u>:

Rule-I : When the equation Mdx + Ndy = 0 is homogeneous and Mx + Ny = 0 then integrating factor of the equation is $\frac{1}{Mx+Ny}$

Problem : Solve the differential equation $\left(x^2y-2xy^2\right)dx-\left(x^3-3x^2y\right)dy=0$

Here $M=x^2y-2xy^2 \;\; and \;\; N=-x^3+3x^2y$

$$\frac{\partial M}{\partial y}=x^2-4xy, \quad \frac{\partial N}{\partial x}=-3x^2+6xy$$

since $\frac{\partial m}{\partial y}\neq\frac{\partial N}{\partial x}\Rightarrow$ Given equation is not exact.

Given equation is homogeneous and

$$Mx+Ny=\left(x^3y-2x^2y^2\right)+\left(-x^3y+3x^2y^2\right)$$

$$=x^2y^2\neq 0$$

So I.F. $=\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$

Multiplying given equation by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

which is exact equation and Its solution is

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

or $\frac{x}{y}-2\log x+3\log y=c$

or $\frac{x}{y}+\log\frac{y^3}{x^2}=c$ is required solution.

Rule-II : If the equation is of the form $f_1(xy).ydx+f_2(xy)xdy=0$ and $Mx-Ny\neq 0$ then the Integrating factor of the equation is $\frac{1}{Mx-Ny}$

Problem : Solve the differential equation $(y-xy^2)dx-(x+x^2y)dy=0$.

Solution : Given differential equation can be written as $(1-xy)ydx-(1+xy)xdy=0$

Here $M=xy-xy^2, N=-x-x^2y$

$$Mx - Ny = \left(xy - x^2y^2\right) - \left(-xy - x^2y^2\right) = 2xy \neq 0$$

So Integrating factor $= \dfrac{1}{Mx - Ny} = \dfrac{1}{2xy}$.

Multiplying Given equation by I.F. we get

$\left(\dfrac{1}{x} - y\right)dx - \left(\dfrac{1}{y} + x\right)dy = 0$ which is exact and its solution is

$$\int\left(\frac{1}{x} - y\right)dx + \int -\frac{1}{y}\,dy = c$$

or $\log x - xy - \log y = c \Rightarrow \log\dfrac{x}{y} - xy = C$.

Rule-III : If the equation $Mdx + Ndy = 0$ *and* $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x say f(x) then I.F. $e^{\int f(x)dx}$.

Problem : Solve the differential equation $\left(x^2 + y^2 + 2x\right)dx + 2ydy = 0$

Solution : $M = x^2 + y^2 + 2x, N = 2$

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2$$

so $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{2y}(2y - 1) = 1$ which is a function of x

I.F. $= e^{\int 1dx} = e^x$

Multiply by e^x we get $e^x\left(x^2 + y^2 + 2x\right)dx + e^x 2ydy = 0$ which is exact and its solution is $\int e^x\left(x^2 + y^2 + 2x\right)dx + \int 0dy = C$.

<u>**Rule - IV**</u> **:** If the equation $Mdx + Ndy = 0$ *and* $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is function of y say f(y) then I.F. $= e^{\int f(y)dy}$.

Problem: Solve the differential equation

$$\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0.$$

Solution : Here $M = 3x^2y^4 + 2xy, N = 2x^2y^3 - x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \quad \frac{\partial N}{\partial x} = 6x^2y^3 - x^2$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{\left(6x^2y^3-x^2\right)-\left(12x^2y^3+2x\right)}{3x^2y^4+2xy}$$

$$=\frac{-2\left(3x^2y^3+2x\right)}{y\left(3x^2y^3+2x\right)}=\frac{-2}{y}$$

I.F. $=e^{\int f(y)dy}=e^{-\int 2/y dy}=e^{-2\log y}=\frac{1}{y^2}$

Multiplying by $\frac{1}{y^2}$, equation becomes

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

which is exact diff equation and its solution is

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx+\int 0dy=C$$

or $x^2+y^2+\frac{x^2}{y}=C$ which is required solution.

<u>**Homogeneous Equation**</u> **:** An equation of the form $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$ is called a homogeneous equation, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of same degree in x and y that is $f_1(x, y) = x^n f_1(y/x)$

$f_2(x, y) = x^n f_2(y/x)$

so $\frac{dy}{dx}=\frac{x^n f_1(x,y)}{x^n f_2(x,y)}=f(y/x)$ .... (1)

Now put y/x $=v$ or $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

So from (1) $v+x\frac{dv}{dx}=f(v)$............ (2)

Separating the variables $\frac{dv}{f(v)-v}=\frac{dx}{x}$

Integrating this, we get required solution.

Problem: Solve the differential equation

$$x^2dy+y(x+y)dx=0$$

Solution : Given equation can be written as $\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$ .... (1)

This is a homogeneous differential equation.

Put y = vx.

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Equation (1) becomes $v+x\frac{dv}{dx}+\frac{vx(x+vx)}{x^2}=0$

or $v+x\frac{dv}{dx}+v(1+v)=0$

or $\frac{dv}{v(v+2)}+\frac{dx}{x}=0$

or $\frac{1}{2}\left[\frac{1}{v}-\frac{1}{v+2}\right]dv+\frac{dx}{x}=0$

Integrating we get $\frac{1}{2}\left[\log v-\log(v+2)\right]+\log x=c$

or $\log x\sqrt{\frac{v}{v+2}}=c$

or $\log x\sqrt{\frac{y/x}{y/x+2}}=c$

or $x\sqrt{\frac{y/x}{y/x+c}}=e^c=k(say)$

or $x^2y=k^2(y+2x)$

Non Homogeneous equation can be reduced to homogeneous form:

Equation of this type is $\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$

Case-I when $\frac{a}{A}\neq\frac{b}{B}$ *then put* $x=X+h$ & $y=Y+k$

so $\frac{dy}{dx}=\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(y+k)+C}=\frac{aX+by+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}$

Choose h and k such that $ah+bk+c=0$ .... (2)

$$Ah+Bk+C=0$$

so equation (2) becomes $\frac{dY}{dX}=\frac{aX+by}{AX+BY}$ which is a homogeneous equation.

and can be solved by putting Y = vX.

Problem : Solve the differential equation $(x+y+5)dy=(y-x+1)dx$

Solution : Given differential equation can be written as $\frac{dy}{dx}=\frac{y-x+1}{x+y+5}$ .... (1)

Put $x=X+h, y=Y+k$

The equation reduced to $\frac{dY}{dX}=\frac{(Y-X)+(k-h+1)}{(X+Y)+(h+k+5)}$ .... (2_)

Now choose h and k such that

k - h + 1 = 0

h + k + 5 = 0

Solving these equations, we get

h = -2, k = -3

so equation (2) becomes $\frac{dY}{dX}=\frac{Y-X}{X+Y}$ .... (3)

Now put Y = vX

$$\Rightarrow \frac{dY}{dX}=v+\frac{dv}{dX}$$

so by (3) $v+x\frac{dv}{dX}=\frac{vX-X}{x+vX}$

or $x\frac{dv}{dX}=\frac{v-1}{v+1}-v$

or $x\frac{dv}{dX}=-\left(\frac{1+v^2}{1+v}\right)$

or $\left(\frac{1+v}{1+v^2}\right)dv=-\frac{dv}{X}$

Integrating $\int\left[\frac{1}{1+v^2}+\frac{2v}{2(1+v^2)}\right]dv+\int\frac{dX}{X}=C$

or $\tan^{-1}v+\frac{1}{2}\log(1+v^2)+\log X=C$

or $\tan^{-1}\frac{Y}{X}+\frac{1}{2}\log\left(\frac{Y^2+X^2}{X^2}\right)+\log X=C$

or $\tan^{-1}\frac{y+3}{x+2}+\frac{1}{2}\log\left[\left(y+3\right)^2+\left(x+2\right)^2\right]=C$

Case-II : When $\frac{a}{A} = \frac{b}{B}$

then let $\frac{a}{A} = \frac{b}{B} = \frac{1}{k} \Rightarrow A = ak \;\; and \;\; B = bk$

so $\frac{dy}{dx} = \frac{ax+by+C}{k(ax+by)+C}$

Now put $ax+by = v \Rightarrow a+b\frac{dy}{dx} = \frac{dv}{dx} \;\; or \;\; \frac{dy}{dx} = \frac{\frac{dv}{dx}-a}{b}$

so $\frac{\frac{dv}{dx}-a}{b} = \frac{v+C}{kv+C} \;\; or \;\; \frac{dv}{dx} = \frac{(b+ak)v+(bc+ac)}{kv+C}$

Example: Solve the differential equation $(4x+6y+5)dy = (3y+2x+4)dx$

Solution : Given Diff. equation can be written as $\frac{dy}{dx} = \frac{3y+2x+4}{4x+6y+5}$

or $\frac{dy}{dx} = \frac{3y+2x+4}{2(2x+3y)+5}$ .... (1)

Put $2x+3y = v \Rightarrow 2+3\frac{dy}{dx} = \frac{dv}{dx}$

So by (1) $\frac{1}{3}\left(\frac{dv}{dx}-2\right) = \frac{v+4}{2v+5}$

or $\frac{dv}{dx} = \frac{7v+22}{2v+5}$

or $\left(\frac{2v+5}{7v+22}\right)dv = dx$

Integrating we get $\int\left[\frac{2}{7}-\frac{9}{7}.\frac{1}{7v+22}\right]dv = \int dx + c$

$$\therefore \frac{2}{7}v - 9\log(7v+22) = x + C$$

$$\therefore \frac{2}{7}(2x+3y) - 9\log(14x+21y+22) = x + C$$

Required solution of given differential equation.

<u>Differential Equation of nth order with constant coefficients</u>:

Linear Differential Equation: If degree of dependent variable and its derivative is one then such differential equation is called linear differential equation.

For example $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=x^3$ is linear differential equation.

Non Linear: If degree of dependent variable and its derivative is more than 1 then such differential equation is called non-linear differential equation.

For example $\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}+y=x^2$ is non-linear differential equation.

General form of a linear differential equation of nth order is

$$\frac{d^ny}{dx^2}+p_1\frac{d^{n-2}y}{dx^{n-2}}+.........+p_ny=X \quad .... (1)$$

where p_1, p_2........p_n are constants and X is any function of X. Operator $\frac{d}{dx}$ is denoted by D.

$$\therefore D^ny+p_1D^{n-1}y+............+p_ny=X \quad .... (2)$$

or $f(D)y=X$

where $f(D)=D^n+p_1D^{n-1}+..........+p_n$

Let us consider a differential equation $\frac{dy}{dx}+py=Q$ of Ist order.

Its solution is $ye^{\int pdx}=\int Qe^{\int pdx}dx+C$

or $\Rightarrow y=Ce^{-\int pdx}+e^{-\int pdx}\int Qe^{\int pdx}dx$

$\Rightarrow y=Cu+v$ *where* $u=e^{-\int pdx}$ & $v=e^{-\int pdx}\int Qe^{\int pdx}dx$

(1) Now differentiating $u=e^{-\int pdx}$ w.r.t. x.

$$\frac{du}{dx}=-pe^{-\int pdx}=-pu\Rightarrow\frac{du}{dx}+pu=0\Rightarrow\frac{d}{dx}Cu+pCu=0$$

so $y=cu$ is the solution of $\frac{d}{dx}(Cu)+p(Cu)=0$.

(2) Differentiating $y=e^{-\int pdx}\int Qe^{\int pdx}dx$ with respect to x.

$$\frac{dv}{dx}=-pe^{-\int pdx}\int Qe^{-\int pdx}dx+e^{-\int pdx}Qe^{-\int pdx}dx$$

$$\Rightarrow\frac{dv}{dx}=-pv+Q\Rightarrow\frac{dv}{dx}+pv=Q$$

so y = v is the solution of $\frac{dv}{dx}+pv=Q$

so solution of (1) is (2) consisting of two partrs i.e. u and v, cu is known as complementary function and v as particular Integral.

So general solution = complementary function + particular Integral.

Method for finding Complementary function:

Let $y=e^{mx}$ *then* $D^r y=m^r e^{mx}$

so equation (2) becomes $\left(m^n+p_1m^{n-1}+p_2m^{n-2}+........+p_n\right)e^{mx}=0$

or $y=e^{mx}$ is a solution of (1) if

$$m^n+p_1m^{n-1}+p_2m^{n-2}+........+p_n=0$$

This equation is known as Auxiliary equation and m_1, m_2, ...m_n are roots of A.E.

There are three cases.

Case-I : Roots are real and different then solution is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+..........+C_ne^{m_nx}$$

Case-II : Roots are real and some of them are equal say $m_1=m_2=m$

then solution is $y=\left(C_1+C_2x\right)e^{mx}+C_3e^{m_3x}+..........+C_ne^{m_nx}$

Case-III : Some of roots are imaginary say $m_1=\alpha+i\beta$, $m_2=\alpha-i\beta$ then general solution of (1) is

$$y=C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x}+C_3e^{m_3x}+........+C_ne^{m_nx}$$

$$=e^{\alpha x}\left[C_1\left(\text{Cos}\,\beta x+i\,\text{Sin}\,\beta x\right)+C_2\left(\text{Cos}\,\beta x-i\,\text{Sin}\,\beta x\right)+......\right]$$

$$=e^{\alpha x}\left[\left(C_1+C_2\right)\text{Cos}\,\beta x+i\left(C_1-C_2\right)\text{Sin}\,\beta x\right]+.............$$

$$=e^{\alpha x}\left[A\,\text{Cos}\,\beta x+B\,\text{Sin}\,\beta x\right]+.............$$

Problem : Solve $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$.

Solution : Given equation is $\left(D^2-8D+15\right)y=0$ so D = 3, 5.

Hence required solution is $y=e^{3x}+C_2e^{5x}$

Problem : $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

$\left(D^2-6D+9\right)y=0$ A.E. is $D^2-6D+9=0$ or $(D-3)^2=0$ or D = 3, 3

Hence required solution is $y=\left(C_1+C_2x\right)e^{3x}$

Problem : Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$

Solution : Here Auxiliary equation is $D^2+4D+5=0$ and its roots are $-2 \pm i$

so required solution is $y=e^{-2x}\left(A\text{Cos}\,x+B\text{Sin}\,x\right)$.

Method of finding particular Integral : Particular Integral of a differential equation $f(D)Y=X$ *is* $\frac{1}{f(D)}X$

Case-I. when $X=e^{ax}$ then $P.I.=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$ *if* $f(a)\neq 0$

If $f(a)=0$ *then* $\frac{1}{f(D)}e^{ax}=x.\frac{1}{f'(D)}e^{ax}$.

Case-II : when $X=x^n$ then P.I. $=\frac{1}{f(D)}x^n=\left[f(D)\right]^{-1}x^n$ expand $\left[f(D)\right]^{-1}$ and then operate.

Case-III : where $X=\text{Sin}\,aX$ then

$$P.I.=\frac{1}{f(D^2)}\text{Sin}\,ax=\frac{1}{f(-a^2)}\text{Sin}\,ax \;\; if \;\; f(-a^2)\neq 0$$

$$and \quad \frac{1}{f(D^2)}\text{Cos}\,ax=\frac{1}{f(-a^2)}\text{Cos}\,ax \;\; if \;\; f(-a^2)\neq 0$$

If $f(-a^2)=0$ *then* $\frac{1}{f(D^2)}\text{Sin}\,ax=x.\frac{1}{f'(-a^2)}\text{Sin}\,ax$

Case-IV : when $X=e^{ax}\phi(x)$

Then P.I. $=\frac{1}{f(D)}e^{ax}.\phi(x)=e^{ax}\frac{1}{f(D+a)}\phi(x)$

Case-V : P.I. = $\frac{1}{D-a}\phi(x)=e^{+ax}\int e^{-ax}.\phi(x)dx$

Case-I.

Problem : (1) Solve the Diff. Equation $\left(D^2+6D+9\right)y=5e^{2x}+e^{-3x}$

Solution : A.E. is $D^2+6D+9=0 \Rightarrow 0=-3,-3$

so C.F. $=\left(C_1+C_2x\right)3e^{-3x}$

and P.I. $=\frac{1}{D^2+6D+9}\left(5e^{2x}+e^{-3x}\right)$

$$= 5\frac{1}{D^2+6D+9}e^{2x} + \frac{1}{D^2+6D+9}e^{-3x}$$

$$= 5\frac{1}{2^2+6.2+9}e^{2x} + x\frac{1}{2D+6}e^{-3x}$$

$$\left(\because D^2+6D+9=0 \;\; at \;\; D=-3\right)$$

$$= \frac{1}{5}e^{2x} + x.x.\frac{1}{2}e^{-3x}$$

$$\left(\because 2D+6=0 \;\; at \;\; D=-3\right)$$

$$= \frac{1}{5}e^{2x} + \frac{x^2}{2}e^{-3x}$$

so general solution is $y = \left(C_1 + C_2 x\right)e^{-3x} + \frac{1}{5}e^{2x} + \frac{x^2}{2}e^{-3x}$.

Case-II.

Problem : Find P.I. to the differential Equation $\left(D^2+4\right)y = x$.

Solution : $P.I. = \frac{1}{D^2+4}x = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x = \frac{1}{4}\left(1-\frac{D^2}{4}+.....\right)x = \frac{x}{4}$

$$= \frac{1}{2D}\left(1-\frac{3}{2}D+\frac{7}{4}D^2+.........\right)x^2$$

$$= \frac{1}{2D}\left(x^2-\frac{3}{2}.2x+\frac{7}{4}.2\right)$$

$$= \frac{1}{2}\left(\frac{x^3}{3}-3\frac{x^2}{2}+\frac{7}{2}x\right)$$

Case-III :

Problem : Solve the differential Equation $\left(D^2-3D+2\right)y = 6e^{3x} + \text{Sin}\,2x$

Solution : A.E. is $m^2-3m+2=0 \Rightarrow m=1, m=2$

so C.F. $= C_1 e^x + C_2 e^{2x}$

P.I. $= \frac{1}{D^2-3D+2}\left(6e^{3x} + \text{Sin}\,2x\right)$

$$= 6\frac{1}{D^2-3D+2}e^{3x} + \frac{1}{D^2-3D+2}\text{Sin}\,2x$$

$$= 6.\frac{1}{2}e^{3x} + \frac{1}{-2^2-3D+2}\text{Sin}\,2x$$

$$= 3e^{3x} - \frac{1}{3D+2}\text{Sin}\,2x$$

$$= 3e^{3x} - \frac{3D-2}{9D^2-4}\text{Sin}\,2x$$

$$= 3e^{3x} - \frac{3D-2}{-40}\text{Sin}\,2x = 3e^{3x} + \frac{1}{40}\left(3\frac{d}{dx} - 2\right)\text{Sin}\,2x$$

$$= 3e^{3x} + \frac{1}{40}\left(3\,\text{Cos}\,2x.2 - 2\,\text{Sin}\,2x\right)$$

$$= 3e^{3x} + \frac{1}{20}\left(3\,\text{Cos}\,2x - \text{Sin}\,2x\right)$$

Hence general solution is $y = C_1e^x + C_2e^{2x} + 3e^{3x} + \frac{1}{20}\left(3\,\text{Cos}\,2x - \text{Sin}\,2x\right)$

Case-IV :

Problem : Solve the differential Equation $\left(D^2 - 4D + 1\right)y = e^{2x}\,\text{Sin}\,2x$

Solution : A.E. is $m^2 - 4m + 1 = 0 \Rightarrow m = 2 \pm \sqrt{3}$

so C.F. $= C_1e^{(2+\sqrt{3})x} + C_2e^{(2-\sqrt{3})x}$

$$= \left(C_1e^{\sqrt{3}x} + C_2e^{-\sqrt{3}x}\right)e^{2x}$$

$$= \left(C_1\,\text{Cosh}\,\sqrt{3}x + C_2\,\text{Sinh}\,\sqrt{3}x\right)e^{2x}$$

P.I. $= \dfrac{1}{D^2 - 4D + 1}e^{2x}\,\text{Sin}\,2x = e^{2x}\dfrac{1}{(D+2)^2 - 4(D+2) + 1}\text{Sin}\,2x$

$$= e^{2x}\frac{1}{D^2 - 3}\text{Sin}\,2x$$

$$= e^{2x}\frac{1}{-4-3}\text{Sin}\,2x$$

$$= -\frac{1}{7}e^{2x}\,\text{Sin}\,2x$$

$$G.S. = \left(C_1\,\text{Cosh}\,\sqrt{3}x + C_2\,\text{Sinh}\,\sqrt{3}x\right)e^{2x} - \frac{1}{7}e^{2x}\,\text{Sin}\,2x$$

Case-V :

Problem : Solve the differential Equation $\left(D^2 - 3D + 2\right)y = e^{5x}$

Solution : $\left(D^2 - 3D + 2\right)y = e^{5x}$

or $\left(D-1\right)\left(D-2\right)y = e^{5x}$

C.F. $y = C_1e^x + C_2e^{2x}$

P.I. $\frac{1}{(D-1)(D-2)}e^{5x} = \left[\frac{1}{(D-2)} - \frac{1}{(D-1)}\right]e^{5x}$

$$= \frac{1}{(D-2)}e^{5x} - \frac{1}{(D-1)}e^{5x}$$

$$= e^{2x}\int e^{5x}.e^{-2x}dx - e^{2x}\int e^{5x}.e^{-x}dx$$

$$= e^{2x}\left(\frac{e^{3x}}{3}\right) - e^{x}\left(\frac{e^{4x}}{4}\right)$$

$$= \frac{e^{5x}}{3} - \frac{e^{5x}}{3} = \frac{e^{5x}}{12}$$

Hence general solution is $y = C_1e^{x} + C_2e^{2x} + \frac{1}{12}e^{5x}$.

Homogeneous Linear Differential Equation with Variable Coefficient (Cauchy-Euler Equation):

A Linear differential equation of type

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_0 y = \phi(x) = Q$$

where a_0, a_1, ... a_n are constants is called a homogeneous linear differential equation with variable coefficient.

To solve this equation put $x = e^z$, $z = \log e^x$ $\frac{d}{dz} = D$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \text{ or } x\frac{dy}{dx} = \frac{dy}{dz} \text{ or } x\frac{dy}{dx} = Dy$$

again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}.\frac{dz}{dx}$

$$= \frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}.\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}\left(D^2 - D\right)y$$

or $$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Similarly $$x^3\frac{d^2y}{dx^3} = D(D-1)(D-2)y$$

Substitution of these values in (1) reduces the given homogeneous equation to a linear differential equation of n^{th} order with constant coefficients.

Working Rule:

Step 1: Write the given equation in D-notation form.

Step II: Replace xD, x^2D^2, x^3D^3 ....etc. in the equation by

$xD = D', x^2D^2 = D'(D'-1), x^3D^3 = D'(D'-1)(D'-2)$ and so on.

Step III: Obtain equations is linear differential equation with constant coefficients, find C.F. and P.I. treating z as independent variable.

Step IV: Lastly put back z = log x to get the required result.

Problem: Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x.$

Solution: D-notation form of the given equation is

$$\left(x^2D^2 - xD + 1\right)y = 2\log x$$

On putting $x = e^z$ and $xD = D', x^2D^2 = D'(D'-1)$, we have

$$\left[D'(D'-1) - D' + 1\right]y = 2z \qquad \left(as \log e^z = z\right)$$

$$\left(D'-1\right)^2 y = 2z$$

Here A.E. is $(m-1)^2 = 0$ gives $m = 1,1.$ $so\ C.F$ is $y = (c_1 + c_2 z)e^z$

Now P.I = $\dfrac{1}{f(D')}X = \dfrac{1}{(D'-1)^2}2z = \left[(D'-1)^{-2}\right]2z$

$$= \left[1 + 2D' + ...\right]2z = 2z + 4$$

Hence the general solution is C.F +P.I. i.e., $y = (c_1 + c_2 z)e^z + 2z + 4$.

$$y = (c_1 + c_2\log x)x + 2\log x + 4 \qquad (putting\ z = \log x) \quad Ans.$$

Problem: solve $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2.$

Solution: D-notation form of the given equation is

$$\left(x^2D^2 - 3xD + 4\right)y = 2x^2$$

On putting $x=e^z$ and $xD=D'$, $x^2D^2=D'(D'-1)$ we have

$$\left[D'(D'-1)-3D'+4\right]y=2e^{2z} \qquad \left(as\, x^2=(e^z)^2\, e^{2z}\right)$$

$$\left(D'^2-4D'+4\right)y=2e^{2z}$$

Here A.E is $(m-2)^2=0$ gives m=2,2, so C.F is $y=(c_1+c_2z)e^{2z}$

Now P.I $=\dfrac{1}{f(D')}X=\dfrac{1}{(D'-1)^2}2e^{2z}=\dfrac{1}{(D'-2)^2}2e^{2z}.1$

$$=2\frac{1}{(D'-2)^2}e^{2z}.1$$

$\left(here\; X\; is\, of\;\; form\, e^{ax}.V, V=1, art.1.2\right)$

$\therefore$ P.I $=2e^{2z}\dfrac{1}{\left\{(D'+2)-2\right\}^2}1=2e^{2z}\dfrac{1}{D'^2}.1 \qquad \left(here\; \dfrac{1}{D'^2}1=\iint 1dzdz\right)$

$$=2e^{2z}\frac{1}{D'}z=2e^{2z}\frac{1}{2}z^2$$

$$=e^{2z}z^2$$

Hence the general solution is i.e., $y=(c_1+c_2z)e^{2z}+e^{2z}z^2$

$$y=(c_1+c_2\log x)x^2+e^{2\log x}(\log x)^2. \qquad (putting\; z=\log x)$$

$$y=(c_1+c_2\log x)x^2+x^2(\log x)^2.$$ Ans.

Problem: Solve $x^3\dfrac{d^3y}{dx^3}-2x^2\dfrac{d^2y}{dx^2}+2y=10\left(x+\dfrac{1}{x}\right).$

Solution: D-notation form of the given equation is

$$\left(x^3D^3+2x^2D^2+2\right)y=10\left(x+\frac{1}{x}\right).$$

On putting x = e^z and xD =D′,x^2 D^2 =D′ (D′-1).X^3 D′ = D′ (D′-1) (D′-2)

We have $\left[D'(D'-1)(D'-2)+2D'(D'-1)+2\right]y=10\left(e^z+e^{-z}\right).$

$$\Rightarrow\left(D'^3+D'^2+2\right)y=10\left(e^z+e^{-z}\right).$$

Here A.E is m^3 -m^2 +2=0 $\Rightarrow$ (m+1) (m^2 -2m+2)= 0

$$\Rightarrow m=-1\, and\; m=\frac{2\pm\sqrt{4-8}}{2}\, or\; m=-1\, and\; m=1\pm i$$

Hence , $C.F = c_1 e^{-z} + e^z\left(c_2 \cos z + c_3 \sin z\right).$

Now P.I = $= \frac{1}{f\left(D'\right)} X = \frac{1}{\left(D'^3 - D'^2 + 2\right)^2} 10\left(e^z + e^{-z}\right).$

$$= 10 \frac{1}{\left(D'^3 - D'^2 + 2\right)^2} e^z + 10 \frac{1}{\left(D'^3 - D'^2 + 2\right)^2} e^{-z}$$

$\therefore$ first part of P.I = $= 10 \frac{1}{\left(1^3 - 1^2 + 2\right)} e^z = 5e^z$ [on putting D′ =1] ....(i)

Now second part of P.I = $10 \frac{1}{\left(D'^3 - D'^2 + 2\right)^2} e^{-z}$

$$= 10 \frac{1}{\left(D'-1\right)\left(D'^2 - 2D' + 2\right)} e^{-z}$$

On putting D′=1 except (D′+1) because it becomes zero, a failure case of f(a) =0

Which can be solved by the formula $\frac{1}{\left(D-\alpha\right)} X = e^{\alpha x} \int e^{-\alpha x} X dx$

$$= 10 \frac{1}{\left(D'+1\right)\left((-1)^2 - 2(-1) + 2\right)} e^{-z}$$

$$= \frac{10}{5\left(D'+1\right)} e^{-z} = 2 \frac{1}{\left(D'+1\right)} e^{-z} = 2e^{-z} \int e^z e^{-z} dz$$

$$= 2ze^{-z}$$ .... (ii)

$\therefore$ from (i) and (ii)

P.I = 5e^z+2+z e^{-z}

Hence the general solution is y = C.F + P.I.= $c_1 e^{-z} + e^z\left(c_2 \cos z + c_3 \sin z\right)$+ 5e^z+2+z e^{-z}

Problem: Solve the differential equation $\left(x^2 D^2 + xD + 4\right) y = 0.$

Solution:

Substituting $x = e^z \Rightarrow In\, x = z \Rightarrow xD = D_1, x^2 D^2 = D_1\left(D_1 - 1\right),$ *the given equation reduces to*

$$\left[D_1\left(D_1 - 1\right) + D_1 - 4\right] y = 0 \Rightarrow \left(D_1^2 - 4\right) y = 0$$

The root of the corresponding characteristic equation are m=2, -2. The required solution of the transformed equation is

$$y = c_1 e^{2z} + c_2 e^{-2z}$$

Putting logx =z, we have the desired solution as

$$y = c_1 x^2 + c_2 x^{-2}$$

Here $c_1 \, and \, c_2$ are arbitrary constants.

Problem: Find the general solution of the differential equation $\left(x^2 D^2 + 1\right) y = 3x^2$.

Solution: substituting $x = e^z$, , the given equation reduces to

$$\left(D_1\left(D_1 - 1\right) + 1\right) y = 3e^{2z} \Rightarrow \left(D_1^{\,2} - D_1 + 1\right) y = 3e^{2z}$$

The characteristic equation of this differential equation is

$$\left(m^2 - m + 1\right) = 0 \Rightarrow m = \left(1 \pm i\sqrt{3}\right)/2$$

The complimentary function is

$$\text{C.F} = e^{\frac{z}{2}}\left[c_1 \cos\left(z\sqrt{\frac{3}{2}}\right) + \left(c_1 \sin z\sqrt{\frac{3}{2}}\right)\right]$$

Substituting z = Inx, we get

$$\text{C.F} = \sqrt{x}\left[c_1 \cos\left(In\, x\sqrt{\frac{3}{2}}\right) + c_1 \sin\left(In\, x\sqrt{\frac{3}{2}}\right)\right]$$

The particular integral of the transformed equation is

$$\text{P.I} = \frac{1}{D_1^{\,2} - D_1 + 1} 3e^{2z} = \frac{1}{2^2 - 2 + 1} 3e^{2z} = e^{2z}$$

Hence the desired solution of the given differential equation is

$$y = \sqrt{x}\left[c_1 \cos\left(In\, x\sqrt{\frac{3}{2}}\right) + c_1 \sin\left(In\, x\sqrt{\frac{3}{2}}\right)\right] + x^2.$$

Legendre 's Linear Differential Equation

The differential equation of the form

$$K_0\left(ax+b\right)^n \frac{d^m y}{dx^n} + K_1\left(ax+b\right)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots.. + K_{n-1}\left(ax+b\right)\frac{dy}{dx} + K_n y = X$$

Where $K_0 K_1 \ldots\ldots\ldots\ldots.., K_n$ are constant and X is a function of x only, known as Legendre' equation. Such equation can be reduced to linear differential equation with constant coefficients by putting

$$\left(ax+b\right) = e^t \; or \; t = \log\left(ax+b\right) \; so\, that \frac{dt}{dx} = \frac{a}{ax+b}.$$

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{dy}{dt}\frac{a}{ax+b}\, or\left(ax+b\right)\frac{dy}{dx}=\left(ax+b\right)\frac{dy}{dt}=a\frac{dy}{dt}=aDy, if\frac{d}{dt}=D$$

Again $$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right)=-\frac{a^2}{\left(ax+b\right)}\frac{dy}{dt}+\frac{a}{ax+b}\frac{d}{dt}\frac{dt}{dx}\left(\frac{dy}{dt}\right)$$

$$=-\frac{a^2}{\left(ax+b\right)^2}\frac{dy}{dt}+\frac{a^2}{\left(ax+b\right)^2}\frac{d^2y}{dt^2}$$

Or $$\left(ax+b\right)^2\frac{d^2y}{dx^2}=a^2\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)=a^2D\left(D-1\right)y\ and\ so\,on$$

By substituting all these values in (2), we obtain linear equation with constant coefficients.

Problem: Solve $\left(1+x\right)^2\frac{d^2y}{dx^2}+\left(1+x\right)\frac{dy}{dx}+y=4\cos\left(\log\left(1+x\right)\right)$

Solution: As the given equation is Legendre's linear equation. Here we take (1+x) = et

$$\left(1+x\right)\frac{d}{dx}y=Dy,$$

$$\left(1+x\right)^2\frac{d^2}{dx^2}y=D\left(D-1\right)y$$

$$\left(1+x\right)^2\frac{d^3}{dx^3}y=\left(D\right)\left(D-1\right)\left(D-2\right)y, ...so\,on$$

, where $D=\frac{d}{dt}$

The given equation reduces to

$$D(D-1)y+Dy+y=4\cos t\Rightarrow\left(D^2+1\right)y=4\cos t, t=\log\left(1+x\right)$$

A.E $\left(D^2+1\right)=0 \quad \Rightarrow D=\pm i$

$\therefore$ C.F $=\left(c_1\cos t+c_2\sin t\right),$

P.I $=\frac{1}{D^2+1}4\cos t=4t\frac{1}{2D}\cos t=2t\frac{D}{D^2}\cos t=2t\frac{D}{-1}\cos t=2t\sin t$

Complete solution, $y=\left(c_1\cos t+c_2\sin t\right)+2t\sin t, t=\log\left(1+x\right).$

Problem: solve $\left(2x+3\right)^2\frac{d^2y}{dx^2}-\left(2x+3\right)\frac{dy}{dx}-12y=6x$

Solution: Take $2x+3=e^t, t=\log\left(2x+3\right)$ so that the given equation reduced to

$$\left[4D(D-1)-2D-12\right]y=3\left(e^t-3\right) as\, 6x=3(2x)=3\left(e^t-3\right)$$

Or $\quad 2\left(2D^2-3D-6\right)y=3\left(e^t-3\right)$

A.E $2D^2-3D-6=0$ or D = $\dfrac{3\pm\sqrt{57}}{4}=m_1,m_2$

$$y_{cf}(t)=c_1e^{m_1t}+c_2e^{m_2t}=c_1\left(e^t\right)^{m_1}+c_2\left(e^t\right)^{m_2}=c_1\left(e^t\right)^{\frac{3\pm\sqrt{57}}{4}}+c_2\left(e^t\right)^{\frac{3\pm\sqrt{57}}{4}}$$

$$P.I=\frac{1}{4D^2-6D-12}3\left(e^t-3\right)$$

$$=3\frac{1}{4D^2-6D-12}e^t-9\frac{1}{4D^2-6D-12}e^{0t}=-\frac{3}{14}e^t+\frac{3}{4}$$

When $y(x)=y_{cf}(x)+y_{PI}(x)=c_1(2x+3)^{m_1}+c_2(2x+3)^{m_2}-\dfrac{3}{14}(2x+3)+\dfrac{3}{4}$

Solution of Second order differential equation:

Equation whose one solution is known:

If y = u is given solution belonging to the complementary of the differential equation. Let other solution be y = v. Then y = uv is complete solution of the differential equation.

Let $\dfrac{d^2y}{dx^2}+p\dfrac{dy}{dx}+Qy=R$ .... (1) be given differential equation and u is the solution included in the complementary function of (1).

So $\dfrac{d^2u}{dx^2}+p\dfrac{du}{dx}+Qy=0$ .... (2)

Now y = uv.

So $\dfrac{dy}{dx}=v.\dfrac{du}{dx}+u.\dfrac{dv}{dx}$ and $\dfrac{d^2y}{dx^2}=v\dfrac{d^2u}{dx^2}+2\dfrac{dv}{dx}\dfrac{du}{dx}+u\dfrac{d^2v}{dx^2}$

Substituting the values of in (1), we get

$$v\frac{d^2u}{dx^2}+2\frac{dv}{dx}\frac{du}{dx}+u\frac{d^2v}{dx^2}+p\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+Qu.v.=R$$

Arranging (Collecting the coefficients of u and v)

$$\Rightarrow v\left(\frac{d^2u}{dx^2}+p\frac{du}{dx}+Qu\right)+u\left(\frac{d^2v}{dx^2}+p\frac{dv}{dx}\right)+2\frac{du}{dx}\frac{dv}{dx}=R$$

The Ist Bracket is zero by virtue of relation (2) and the remaining is divided by u.

$$\frac{d^2v}{dx^2}+\left[p+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{R}{u} \text{ .... (3)}$$

Let $\frac{dv}{dx}=z$ so that $\frac{d^2u}{dx^2}=\frac{dz}{dx}$

So equation (3) becomes $\frac{dz}{dx}+\left[p+\frac{2}{u}\frac{du}{dx}\right]z=\frac{R}{u}$

This is a linear differential equation which can be solved (z can be found) which contains one constant on integration.

$z=\frac{dv}{dx}$, we can get v. So the solution is y = uv.

Rules to find out the integral belonging to the complementary function.

Criteria	Part of C.F.
1. $1+P+Q=0$	e^x
2. $1-P+Q=0$	e^{-x}
3. $1+\frac{P}{a}+\frac{Q}{a^2}=0$	e^{ax}
4. $P+Qx=0$	x
5. $2+Px+Qx^2=0$	x^2
6. $n(n-1)+Pnx+Qx^2=0$	x^n

Problem : Solve $y''-4xy'+\left(4x^2-2\right)y=0$ given that $y=e^{x^2}$ is an integral included in the complementary function.

Solution : $y''-4xy'+\left(4x^2-2\right)y=0$ .... (1)

On putting $y=ve^{x^2}$, the reduced equation is

$$\frac{d^2v}{dx^2}+\left[p+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=0$$

$$\Rightarrow\frac{d^2v}{dx^2}+\left[-4x+\frac{2}{e^{x^2}}\left(2xe^{x^2}\right)\right]\frac{dv}{dx}=0$$

$$\Rightarrow\frac{d^2v}{dx^2}+\left[-4x+4x\right]\frac{dv}{dx}=0\Rightarrow\frac{d^2v}{dx^2}=0\Rightarrow\frac{dv}{dx}=c\Rightarrow v=C_1x+C_2$$

so complete solution is $\therefore\ y = uv = e^{x^2}\left(C_1 x + C_2\right)$

Problem : Solve the differential equation $x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3$

Solution : $x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3$ .....(1)

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2x(1+x)}{x^2}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$$

here $P + Qx = \dfrac{-2x(1+x)}{x^2} + \dfrac{2(1+x)}{x^2}x = 0$

Hence $y = x$ is a solution of the complementary function and let other solution is v.

Putting y = vx in (1), we get reduced equation

$$\frac{d^2v}{dx^2} + \left\{P + \frac{2}{u}\frac{du}{dx}\right\}\frac{dv}{dx} = \frac{x}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{-2x(1+x)}{x^2} + \frac{2}{x}.(1)\right]\frac{dv}{dx} = \frac{x}{x}$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \Rightarrow \frac{dz}{dx} - 2z = 1 \ \ as \ \ \frac{dv}{dx} = z$$

which is a linear differential equation of Ist order

and I.F. = $e^{\int -2dx} = e^{-2x}$

and its solution is $z\,e^{-2x} = \int e^{-2x}dx + C_1$

$$ze^{-2x} = \frac{e^{-2x}}{-2} + C_1 \ \ or \ \ z = -\frac{1}{2} + C_1e^{2x}$$

$$\Rightarrow \frac{dv}{dx} = -\frac{1}{2} + C_1e^{2x}$$

or $dv = \left(-\dfrac{1}{2} + C_1e^{2x}\right)dx \Rightarrow v = -\dfrac{x}{2} + \dfrac{C_1}{2}e^{2x} + C_2$

so $y = uv = x\left(-\dfrac{x}{2} + \dfrac{C_1}{2}e^{2x} + C_2\right)$

Problem : Solve $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$

Solution : $\dfrac{d^2y}{dx^2} - \dfrac{(2x+5)}{(x+2)}\dfrac{dy}{dx} + \dfrac{2y}{(x+2)} = \dfrac{(x+1)e^x}{(x+2)}$ ......(1)

Here $1+\frac{p}{a}+\frac{Q}{a^2}=0$ choosing a = 2

$$1-\frac{2x+5}{2x+4}+\frac{2}{4x+8}=0$$

Hence $y=e^{2x}$ is a part of C.F.

Putting $y=e^{2x}v$ in (1), the reduced equation is

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{(x+1)e^{x}}{e^{2x}(x+2)}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{2x+5}{x+2}+\frac{2}{e^{2x}}.2e^{2x}\right]\frac{dv}{dx}=\frac{(x+1)e^{x}}{e^{2x}(x+2)}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{2x+5}{x+2}+4\right]\frac{dv}{dx}=\frac{(x+1)e^{-x}}{(x+2)}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\frac{2x+3}{x+2}\frac{dv}{dx}=\frac{(x+1)e^{-x}}{(x+2)}$$

$$\Rightarrow \frac{dz}{dx}+\frac{2x+3}{x+2}z=\frac{(x+1)e^{-x}}{(x+2)} \quad \left(\frac{dv}{dx}=z\right)$$

which is a linear differential equation with

I.F. = $e^{\int\frac{2x+3}{x+2}dx}=e^{\int\left(2-\frac{1}{x+2}\right)dx}=e^{2x-\log(x+2)}=\frac{e^{2x}}{(x+2)}$

Its Solution is $z.\frac{e^{2x}}{(x+2)}=\int\frac{e^{2x}}{x+2}\frac{x+1}{(x+2)}e^{-x}dx+c$

$$=\int\frac{e^{x}(x+1)}{(x+2)^2}dx+c=\int e^{x}\left(\frac{1}{(x+2)}-\frac{1}{(x+2)^2}\right)dx+c$$

$$\Rightarrow z=e^{-x}+C(x+2)e^{-2x}$$

$$\Rightarrow \frac{dv}{dx}=e^{-x}+C(x+2)e^{-2x}$$

$$\Rightarrow v=\int e^{-x}dx+C\int(x+2)e^{-2x}dx+C_1$$

so $y=uv$

$$=e^{2x}\left[-e^{-x}+\frac{Ce^{-2x}}{4}(2x+5)+C_1\right]$$

Normal form (Removal of First derivative)

Consider the differential equation $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+Qy=R$ ......(1)

Let y = uv be the complete solution of equation (1).

so $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ .... (1)

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

putting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1) we get

$$\left(u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}\right) + p\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

$$v\frac{d^2u}{dx^2} + \frac{du}{dx}\left(pv + 2\frac{dv}{dx}\right) + u\left(\frac{d^2u}{dx^2} + p\frac{dv}{dx} + Qv\right) = R$$

$$\frac{d^2u}{dx^2} + \frac{du}{dx}\left(p + \frac{2}{v}\frac{dv}{dx}\right) + \frac{u}{v}\left(\frac{d^2u}{dx^2} + p\frac{dv}{dx} + Qv\right) = \frac{R}{v}$$

Last bracket is not zero as y = v is not part of complementary function.

Now removing the Ist derivative $P + \frac{2}{v}\frac{dv}{dx} = 0 \quad or \quad \frac{dv}{v} = -\frac{1}{2}Pdx$

or log $v = -\frac{1}{2}\int Pdx \Rightarrow v = e^{-\frac{1}{2}\int Pdx}$

Now our objective is to find the value of last bracket

i.e. $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = R$

Now $\frac{dv}{dx} = -\frac{1}{z}e^{-\frac{1}{2}\int pdx} = -\frac{1}{2}pv \quad as \quad v = e^{-\frac{1}{2}\int pdx}$

$$\frac{d^2v}{dx^2} = -\frac{1}{2}\frac{dp}{dx}v - \frac{p}{2}\frac{dv}{dx} = -\frac{1}{2}\frac{dp}{dx}v - \frac{p}{2}\left(-\frac{1}{2}pv\right)$$

$$= -\frac{1}{2}\frac{dp}{dx}v + \frac{1}{u}p^2v$$

$$\therefore \frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qv = -\frac{1}{2}\frac{dp}{dx}v + \frac{1}{4}p^2v + p\left(-\frac{1}{2}pv\right) + Q$$

$$= v\left(Q - \frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2\right)$$

so equation (1) is transformed as

$$\frac{d^2u}{dx^2} + \frac{u}{v}v\left(Q - \frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2\right) = \frac{R}{v}$$

$$\Rightarrow \frac{d^2u}{dx^2} + u\left(Q - \frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2\right) = \mathrm{Re}^{\frac{1}{2}\int pdx}$$

$$\text{or } \frac{d^2u}{dx^2} + Q_1 u = R_1 \quad \text{where } Q_1 = Q - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4}$$

$$R_1 = \mathrm{Re}^{\frac{1}{2}\int \phi dx}$$

so $y = uv \,\&\, v = \mathrm{e}^{-\frac{1}{2}\int pdx}$.

Working Rule to solve linear second order differential equations by reducing to its normal form:

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \textit{coefficient of } \frac{d^2y}{dx^2} \textit{ is unity.}$

Step 2: Find v = $e^{-\frac{1}{2}\int Pdx}, Q_1 = \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right) \textit{and } R_1 = R.e^{\frac{1}{2}\int Pdx} = \frac{R}{v}.$

Step 3: Put the values of Q_1 and R_1 in normal form $\frac{d^2u}{dx^2} + Q_1 u = R_1.$

Step 4: Obtained equation is linear differential equation with constant coefficient and solve by finding C.F and P.I

Step 5: Required solution is obtain by putting the value of v and u in y = u v.

Problem Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2 y = e^{x^2}$

Solution: Here P = -4 x ,Q = $4x^2$, R = e^{x^2} to reduce in normal form we choose

$$v = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}, R_1 = \frac{R}{v} = \frac{e^{x^2}}{e^{x^2}} = 1$$

And $Q_1 = \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right) = \left(4x^2 - \frac{1}{4}(-4x)^2 - \frac{1}{2}\frac{d(-4x)}{dx}\right) = 2$

$\therefore$ Equation reduces to $\frac{d^2u}{dx^2} + 2u = 1 \, or \left(D^2 + 2\right)u = 1.$

Here A.E is $m^2 + 2 = 0 \Rightarrow m = \pm\sqrt{2i} \;\therefore C.F = c_1 \cos\sqrt{2x} + c_2 \sin\sqrt{2x}$

And P.I = $\frac{1}{f(D)}R = \frac{1}{\left(D^2+2\right)}1 = \frac{1}{2}$ therefore u = C.F + P.I = $c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2},$

And complete solution is y = u $y_1 = \left(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2}\right)e^{x^2}.$ *Ans.*

Problem : Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\,\mathrm{Sin}\,2x$

Solution : We have $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)y=-3e^{x^2}\,\text{Sin}\,2x$ .... (1)

here $p=-4x,\;\; Q=4x^2-1,\;\; R=-3e^{x^2}\,\text{Sin}\,2x$

In order to remove Ist derivative, $v=e^{-\frac{1}{2}\int pdx}=e^{-\frac{1}{2}\int -4xdx}=e^{2\int xdx}=e^{x^2}$

On putting $y = uv$, the normal equation is

$$\frac{d^2u}{dx^2}+Q_1u=R_1 \quad .... (2)$$

$$\text{where } \; Q_1=Q-\frac{1}{2}\frac{dp}{dx}-\frac{p^2}{4}$$

$$R_1=\frac{R}{v}=\text{Re}^{\frac{1}{2}\int Pdx}$$

$$Q_1=Q-\frac{1}{2}\frac{dp}{dx}-\frac{p^2}{x}=\left(4x^2-1\right)-\frac{1}{2}(-4)-\frac{16x^2}{4}=4x^2-1+2-4x^2=1$$

$$R_1=\frac{R}{v}=\frac{-3e^{x^2}\,\text{Sin}\,2x}{e^{x^2}}=-3\sin 2x$$

Auxiliary Equation is $m^2+1=0\Rightarrow m=\pm i$

Hence C.F. = $c_1\cos x+c_2\sin x$

P.I.= $\frac{1}{D^2+1}(-3\sin 2x)=\frac{-3}{-4+1}\sin 2x=\sin 2x$

So solution is $v=c_1\cos x+c_2\sin x+\sin 2x$

Hence complete solution of given differential equation is $y=uv=e^{x^2}(c_1\cos x+c_2\sin x+\sin 2x).$

Solution of the $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$ by changing the independent variable:

Sometimes the equation is transformed into an integrable form by changing the independent variable. Let the equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X \qquad \text{...... (1)}$$

Let the independent variable x be changed to z by taking z as the function of x.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}\, and\, \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right)=\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\frac{d^2z}{dx^2},$$

Substituting these value in (1), we get

$$\left[\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2\right]+\frac{d^2z}{dx^2}+P\left(\frac{dy}{dz}\frac{dz}{dx}\right)+Qy=X\ ,$$

Or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\left[\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right]\frac{dy}{dz}+Qy=X\ ,$$

Or $$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1,$$

Where $P=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2} and X_1=\frac{X}{\left(\frac{dz}{dx}\right)^2}$

After obtaining equation (2) we like to choose z in such a way that (2) can be easily integrating.

Case 1: $P_1 = 0$

We choose z to make the coefficient of $\frac{dy}{dz}$ in (2), equal to zero i.e.

$$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=0 \text{ or } \frac{d^2z}{dx^2}+P\frac{dz}{dx}=0\, or\, \frac{\frac{d^2z}{dx^2}}{\frac{dz}{dx}}=-P$$

Integrating, we get $\log\frac{dz}{dx}=-\int Pdx\, or\, \frac{dz}{dx}=e^{-\int Pdx}$

Integrating again, we get z = $\int e^{-\int Pdx}dx,$ this value of x reduce (2) to $\frac{d^2y}{dz^2}+Q_1y=X_1,$

Which can be easily solved provided Q_1 comes out to be a constant or a constant multiplied by $\frac{1}{z^2}$.

Case 2: $Q_1 = a^2$

We choose z such that $Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=a^2$ (constant),

i.e. $a^2.\left(\frac{dz}{dx}\right)^2=Q\, or\, a\frac{dz}{dx}=\sqrt{Q},$ integrating gives $az=\int\sqrt{Q}dx.$

the above value of z reduces (2) to

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+a^2y+X_1,$$

Which can be solved easily, if P_1 comes out to be a constant.

Problem: Solve $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x.$

Solution: writing given equation in standard form, we have

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^4 x. \qquad \text{......(i)}$$

Here $P = \tan x, Q = -2\cos^2 x, X = 2\cos^4 x$

Changing independent variable from x to z, equation becomes,

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + a^2 y = X_{1,} \qquad \text{.... (ii)}$$

Where, $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ $and\ X_1 = \dfrac{X}{\left(\frac{dz}{dx}\right)^2}$

Let us choose z such that $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ = -2 (constant)

i.e., $-2.\left(\frac{dz}{dx}\right)^2 = -2\cos^2\ or\ \frac{dz}{dx} = \cos x,$ integrating gives z=sin x. .... (iii)

then $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ = $\dfrac{-\sin x + \tan x(\cos x)}{(\cos x)^2} = 0$ and $X_1 = \dfrac{X}{\left(\frac{dz}{dx}\right)^2} = \dfrac{2\cos^4 x}{(\cos)^2} = 2\cos^2 x$

Hence equation (ii) transferred to

$$\frac{d^2y}{dz^2} - 2y = 2\cos^2 x, or \frac{d^2y}{dz^2} - 2y = 2(1-\sin^2 x), or \frac{d^2y}{dz^2} - 2y = 2(1-z^2),$$

Or $(D^2 - 2)y = 2(1-z^2),$

$$Now\ C.F = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$$

$$P.I = \frac{1}{(D^2-2)} 2(1-z^2) = \frac{2}{-2}\left(1 - \frac{D^2}{2}\right)^{-1}(1-z^2) = -\left(1 + \frac{D^2}{2}\right)(1-z^2) = z^2,$$

Hence the solution of the given equation is

$$y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

$$or\ y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + (\sin x)^2 \quad [As\ z = \sin x]$$ Ans.

Problem: Solve $\left(1+x^2\right)^2 \frac{d^2y}{dx^2} + 2x\left(1+x^2\right)\frac{dy}{dx} + 4y = 0.$

Solution: writing equation in standard form, we have

$$\frac{d^2y}{dx^2} + \frac{2x}{\left(1+x^2\right)}\frac{dy}{dx} + \frac{4}{\left(1+x^2\right)^2} y = 0. \qquad \text{......(i)}$$

Here $P = \frac{2x}{\left(1+x^2\right)}, Q = \frac{4}{\left(1+x^2\right)^2}, X = 0.$

Changing independent variable from x to z, equation becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = X_1. \qquad \text{......(ii)}$$

Where, $P_1 = \frac{\frac{d^2z}{dx^2} + p\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \; and \; X_1 = \frac{X}{\left(\frac{dz}{dz}\right)^2}.$

Let us choose z such that $P_1 = 0$ i.e $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0.$

A.T.Q. $\frac{d^2z}{dx^2} + \frac{2x}{\left(1+x^2\right)}\frac{dz}{dx} = 0 \; Putting \; \frac{dz}{dx} = t \; then \; \frac{dt}{dx} + \frac{2x}{\left(1+x^2\right)}\frac{dz}{dx} = 0$

Separating the variable and integrating, we get

$$\log t + \log\left(1+x^2\right) = 0 \; or \; t = \frac{1}{\left(1+x^2\right)}$$

Since $\frac{dz}{dx} = t$ gives $\frac{dz}{dx} = \frac{1}{\left(1+x^2\right)}$ separating the variable and integrating, $z = \tan^{-1} x$

$$\therefore \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{4}{\left(1+x^2\right)^2}}{\left(\frac{1}{\left(1+x^2\right)}\right)^2} = 4 \; and \; X_1 = \frac{0}{\left(\frac{1}{\left(1+x^2\right)}\right)^2} = 0,$$

Hence the transformed equation is

$$\frac{d^2y}{dz^2} + 4y = 0.$$

Its solution is $y = c_1 \cos 2z + c_2 \sin 2z$

Or $\qquad y = c_1 \cos 2\left(\tan^{-1} x\right) + c_2 \sin 2\left(\tan^{-1} x\right) \qquad \left[As \;\; z = \tan^{-1} x\right]$ Ans.

Simultaneous Differential Equation:

If two or more dependent variables are functions of a single independent variable, then the equations involving their derivatives are called simultaneous equations. For ex. $\frac{dx}{dt}+4y=t$

$$\frac{dy}{dt}+2x=e^t$$

Method of solving these equations is based on the process of elimination as we solve algebraic simultaneous equations.

Problem: solve $\frac{dx}{dy}+2y=e^t, \frac{dy}{dt}-2x=e^{-t}$

Solution: The given equation in symbolic form can be written as

$$Dx+2y=e^t \qquad \text{... (1)}$$

$$Dy-2x=e^{-t} \qquad \text{... (2)}$$

Operate D on (2) and add to it 2 time of (1), we get

$$\left(D^2+4\right)y=2e^t-e^{-t} \qquad \text{... (3)}$$

Here A.E is $D^2+4=0$ $i.e$ $D=\pm 2i$

$$\therefore \quad y_{C.F}(t)=\left(c_1\cos 2t+c_2\sin 2t\right) \qquad \text{...(4)}$$

And $y_{P.I}(t)=\frac{1}{D^2+4}\left(2e^t-e^{-t}\right)=2\frac{1}{D^2+4}e^t-\frac{1}{D^2+4}e^{-t}=2\frac{e^t}{5}-\frac{e^{-t}}{5}$ ...(5)

Hence $y(t)=\left(c_1\cos 2t+c_2\sin 2t\right)+\frac{2e^t}{5}-\frac{e^{-t}}{5}$ ...(6)

Now from (2), $x=\frac{1}{2}\left[Dy-e^{-t}\right]=\frac{1}{2}\left[D\left(c_1\cos 2t+c_2\sin 2t\right)+D\left(\frac{2}{5}e^t+\frac{e^{-t}}{5}-e^{-t}\right)\right]$

$$=\frac{1}{2}\left(-2c_1\sin 2t+2c_2\cos 2t+\frac{2e^t}{5}+\frac{e^{-t}}{5}-e^{-t}\right)$$

$$=-c_1\sin 2t+c_2\cos 2t+\frac{e^t}{5}-\frac{2}{5}e^{-t} \qquad \text{... (7)}$$

Problem: solve $(D-1)x+Dy=2t+1,(2D+1)x+2Dy=t$

Solution: For elimination of y, take difference of 2 time of 1st from 2nd i.e.,

$$\left(\left(2D+1\right)x+2Dy\right)-2\left(\left(D-1\right)x+Dy\right)=t-2\left(2t+1\right)$$

Or $$3x=-3t-2\, or\, x(t)=-t-\frac{2}{3} implying\, \frac{dx}{dt}=-1$$

Now using above values of $x(t)$ and $\frac{dx}{dt}$ in 1ST Equation, we get

$$Dx-x+Dy=2t+1 or -1-\left(-t-\frac{2}{3}\right)+Dy=2t+1$$

Implying $$Dy=t+\frac{4}{3} \quad i.e \quad y(t)=\frac{t^2}{2}+\frac{4}{3}t+c$$

Where c is a constant of integration.

Problem: solve $\frac{dx}{dt}-7x+y=0, \frac{dy}{dt}-2x-5y=0$

Solution: The given equation in symbolic form are written as:

$$\left(D-7\right)x+y=0 \qquad \text{... (1)}$$

$$-2x+\left(D-5\right)y=0 \qquad \text{... (2)}$$

To eliminate y, operate $\left(D-5\right)$ on (1) and add the two equations to get

$$\left(D-5\right)\left(D-7\right)x+2x=0 \text{ or } \left(D^2-12D+37\right)x=0 \qquad \text{...(3)}$$

So that A.E is $\left(D^2-12D+37\right)$ = 0 or $D=6\pm i$

$$\therefore \qquad x_{CF}(t)=e^{6t}\left(c_1\cos t+c_2\sin t\right) \qquad \text{... (4)}$$

Implying $$\frac{dx}{dt}=e^{6t}\left(-c_1\cos t+c_2\sin t\right)+6e^{6t}\left(c_1\cos t+c_2\sin t\right) \qquad \text{...(5)}$$

On substituting value of x(t) and Dx from (4) and (5) respectively in equation (1), we get

$$e^{6t}\left(-c_1\cos t+c_2\sin t\right)+6e^{6t}\left(c_1\cos t+c_2\sin t\right)-7e^{6t}\left(c_1\cos t+c_2\sin t\right)+y=0$$

Or $$y=e^{6t}\left[\left(c_1-c_2\right)\cos t+\left(c_1+c_2\right)\sin t\right]$$

$$=e^{6t}\left[C\cos t+D\sin t\right]$$

Where, $C=c_1-c_2\, and\, D=c_1+c_2$ ...(6)

<u>Method of Variation of Para Meters</u>: Method of variation of parameters is the method of finding the general solution of any second order non homogeneous linear differential equation both for variable and constant coefficients whose complementary function is known.

Step for solution:

1. Find out the parts of C.F.
2. Let the y_1 and y_2 be parts of complementary function.
3. Consider y = y_1 u + y_2 v as the complete solution of equation given
4. A and B are determined by the formula

$$u = -\int \frac{-y_2 R}{y_1 y_2' - y_1' y_2} dx + c_1 \; and \; v = \int \frac{y_2 R}{y_1 y_2' - y_1' y_2} dx + c_2$$

Where c_1 and c_2 are the arbitrary constants of integration.

4. Put the values of u and v in y = u y_1 + v y_2 to get the complete solution.

Problem: By method of variation of parameters solve the following Differential equation y'' + y = Sec x.

Solution : we have $\frac{d^2y}{dx^2} + y = \text{Sec}\, x$ .... (1)

A.E. is $\left(D^2+1\right) = 0 \Rightarrow m = \pm i$

So C.F. = c_1 Cos x + c_2 Sin x

Here y_1 = Cos x, y_2 = Sin x

So let complete solution of (1) is

C.S. = $u\cos x + v\sin x$

Let complete solution be $y = uy_1 + vy_2 = u\cos x + v\sin x$ .... (2)

where

$$u = \int \frac{-y_2 R}{y_1 y_2' - y_1' y_2} dx = -\int \frac{\sin x}{\{\cos x \times \cos x - (-\sin x)\sin x\}} \times \sec x dx = -\int \tan x\, dx + c_1 = \log\cos x + c_1$$

$$\& v = \int \frac{y_2 R}{y_1 y_2' - y_1' y_2} dx = \int \frac{\cos x}{\{\cos x \times \cos x - (-\sin x)\sin x\}} \times \sec x\, dx = \int dx + c_2 = x + c_2$$

hence complete solution y = $u\cos x + v\sin x = (\log\cos x + c_1)\cos x + (x + c_2)\sin x.$

Problem : Solve the following differential equation by method of variation of parameters $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Solution : we have $\left(D^2-1\right)y = \frac{2}{1+e^x}$

A.E. is $\left(D^2-1\right) = 0 \Rightarrow m = \pm 1$

So C.F. $=C_1e^{x}+C_2e^{-x}$

Let P.I. $=uy_1+vy_2=ue^{x}+ve^{-x}$ *and* W= $y_1y_2'-y_1'y_2=-e^{x}e^{-x}-e^{x}e^{-x}$

$$u=\int\frac{-y_2R}{y_1y_2'-y_1'y_2}dx=-\int\frac{e^{-x}}{-2}\times\frac{2}{1+e^{x}}dx=\int\frac{e^{-x}}{1+e^{x}}dx$$

$$\int\frac{1}{e^{x}\left(1+e^{x}\right)}dx=\int\left(\frac{1}{e^{x}}-\frac{1}{e^{x}+1}\right)dx$$

$$=\int e^{x}dx-\int\frac{e^{-x}}{1+e^{-x}}dx$$

$$=-e^{-x}+\log\left(e^{-x}+1\right)$$

$$\&v=\int\frac{y_1R}{y_1y_2'-y_1'y_2}dx=\int\frac{e^{x}}{-2}.\frac{2}{1+e^{x}}dx=\int\frac{e^{x}}{1+e^{x}}dx=-\log(1+e^{x})$$

P.I. $=uy_1+vy_2=\left[-e^{-x}+\log\left(e^{-x}+1\right)\right]e^{x}-e^{-x}\log(1+e^{x})$

$$=-1+e^{x}\log\left(e^{-x}+1\right)-e^{-x}\log(e^{x}+1)$$

So, $y=C_1e^{x}+C_2e^{-x}-1+e^{x}\log\left(e^{-x}+1\right)-e^{-x}\log(e^{x}+1)$

<u>Solution of Second order differential equation by changing in dependent variable</u>:

Consider second order linear differential equation.

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

Step for solution:

1. Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare the equation with standard from $y''+Py'+Qy=R$ and get P,Q, and R
3. Choose z such that $\left(\frac{dz}{dx}\right)^2=Q$

Here Q is taken in such a way that it remains the whole square of a function without surd and its negative sign is ignored.

4. Find $\frac{dz}{dx}$ hence obtain z (on integration) and $\frac{d^2z}{dx^2}$ (on differentiation)
5. Find P_1, Q_1 and R_1 by the formulae

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

6. Reduced equation is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in term of z.
7. We write the complete solution as y in term of x by replacing the value of z in term of x.

Example: By changing the independent variable solve the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4$$

Solution: $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4$ ...(1)

Here, $P = -\frac{1}{x}, Q = 4x^2, R = x^4$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow$ $\frac{dz}{dx} = 2x$ ...(2)

$z=x^2$ (on integration)

from (2), $\frac{d^2z}{dx^2} = 2$ (Differentiating (2) w.r.t x)

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{4x^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^4} = \frac{x^2}{4}$$

Reduced equation is

$$\frac{d^2y}{dx^2} + y = \frac{z}{4} \qquad \left[\therefore z = x^2 \; from\,(3)\right]$$

Auxiliary equation is,

$$m^2+1=0 \qquad \Rightarrow \qquad m=\pm i$$

$$C.F=c_1\cos z+c_2\sin z$$

$$P.I=\frac{1}{D^2+1}\left(\frac{z}{4}\right)=\left(1+D^2\right)^{-1}\left(\frac{z}{4}\right)$$

$$= \ (1-D^2+..............)\left(\frac{z}{4}\right)=\frac{z}{4} \qquad \text{(Leaving higher powers)}$$

$\therefore$ solution is $y=c_1\cos z+c_2\sin z+\dfrac{z}{4}$

Complete solution is given by

$$y=c_1\cos(x^2)+c_2\sin(x^2)+\frac{x^2}{4}$$

Example: By changing the independent variable solve the differential equation $\left(1+x\right)^2\dfrac{d^2y}{dx^2}+\left(1+x\right)\dfrac{dy}{dx}+y=4\cos\log\left(1+x\right)$

Solution: $$\frac{d^2y}{dx^2}+\frac{1}{(1+x)}\frac{dy}{dx}+\frac{y}{\left(1+x\right)^2}=\frac{4}{\left(1+x\right)^2}\cos\log\left(1+x\right) \qquad ...(1)$$

Choose z such that,

$$\left(\frac{dz}{dx}\right)^2=\frac{1}{\left(1+x\right)^2}$$

$$\Rightarrow \qquad \frac{dz}{dx}=\frac{1}{1+x} \qquad ...\ (2)$$

Integration yields, z = log(1+x) ... (3)

From (2), $\dfrac{d^2z}{dx^2}=-\dfrac{1}{\left(1+x\right)^2}$

$$P_1=\frac{\dfrac{1}{\left(1+x\right)^2}+\dfrac{1}{\left(1+x\right)}+\dfrac{1}{\left(1+x\right)}}{\dfrac{1}{\left(1+x\right)^2}}=0$$

$$Q_1=\frac{Q}{\left(\dfrac{dz}{dx}\right)^2}=1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)} = 4\cos\log(1+x) = 4\cos z \qquad \text{from (3)}$$

Reduced equation is

$$\frac{d^2y}{dx^2} + y = 4\cos z$$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F = c_1 \cos z + c_2 \sin z$$

$$P.I = \frac{1}{D^2+1}(4\cos z) = 4, \frac{z}{2}\sin z = 2z\sin z$$

Complete solution is

$$y = c_1 \cos z + c_2 \sin z + 2z\sin z$$

$$y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x)\sin\log(1+x) .$$

Series solution of second order ODEs

Ordinary and singular points:

Consider the second order differential equation of the form,

$$\begin{aligned}
& y'' + P(X)y' + Q(x)y = 0 \\
& xP(X),\ x^2Q(x) \textit{ are Analytic } x = 0 \\
& \Rightarrow P(X), Q(x) \textit{ are not Analytic } x = 0 \\
& \Rightarrow x = 0 \textit{ is a regular singulanor point}
\end{aligned}$$

Ordinary point: A point $x=x_0$ is called an ordinary point of the equation Eq. (1) if both P(x), Q(x) are analytic at $x=x_0$

singular point: if the point $x=x_0$ is not an ordinary point of the equation Eq (1), then it is called a singular point. There are two types of singular points.

Regular singular point: A singular point $x=x_0$ is called regular singular point of the equation Eq(1) if both $(x-x_0)P(x), (x-x_0)^2 Q(x)$ are analytic at $x=x_0$

Irregular singular point: A singular point, which is not regular is called irregular singular point.

Example verify that origin is an ordinary point of regular singular point of the equations

1. $Y'' + xy' + y = 0$
2. $2x^2 Y'' + xy' - (x+1) y = 0$

Example- verify that origin is an ordinary point or singular point of the equations.

1. $y'' + xy' + y = 0$

2. $2x^2y'' + xy' - (x+1)y = 0$

Solution:

1. The equation is $y'' + xy' + y = 0$

Compare this with $y'' + P(X)y' + Q(x)y = 0$

Therefore $P(X) = \frac{1}{2x^2}, Q(x) = -\frac{(x+1)}{2x^2}$

$\Rightarrow At\, x = 0, P(X)\, and\, Q(x)\, are\, defined.$

$\Rightarrow P(X), Q(x)\, are\ Analytic\ x = 0$

$\Rightarrow x = 0\, is\, ordinary\ po\mathrm{int}$

2. Dividing the equation by $2x^2$ i.e., $y'' + \frac{y'}{2x^2} - \frac{(x+1)}{2x^2}y = 0$

Compare this with $y'' + P(X)y' + Q(x)y = 0$

Therefore $P(X) = \frac{1}{2x^2}, Q(x) = -\frac{(x+1)}{2x^2}$

At x=0, P(x), Q(x) are not defined.

$\Rightarrow P(X), Q(x)\, are\ not\ Analytic\ x = 0$

$\Rightarrow x = 0\, is\, not\, an\, ordinary\ point$

$\Rightarrow x = 0\, is\, a\, singular\ point$

Now, $xP(X) = \frac{1}{2},\ x^2Q(x) = -\frac{(x+1)}{2}$

$xP(X),\ x^2Q(x)\, are\ \ Analytic\ x = 0$

$\Rightarrow x = 0\, is\, a\, regular\, singulanor\ point$

Example: verify that $x = 1$ is a regular point of the equation

$$(x^2 - 1)y'' + xy' - y = 0$$

Solution: Dividing the equation by $(x^2 - 1)$ i.e.

$$y'' + \frac{x}{(x^2-1)}y' - \frac{1}{(x^2-1)}y = 0$$

Compare this with $y'' + P(X)y' + Q(x)y = 0$

Therefore $p(x) = \frac{x}{(x^2-1)}, Q(x) = -\frac{1}{(x^2-1)}$

$\Rightarrow At\ x=1, p(x), Q(x)$ *are not defined.*

$\Rightarrow p(x), Q(x)$ *are not analytic* $x=1$

$\Rightarrow x=1$ *is not an ordinary point*

$\Rightarrow x=1$ *is a singular point*

Now

$$(x-1)P(x)=(x-1)\frac{x}{(x^2-1)}=\frac{x(x-1)}{(x-1)(x+1)}=\frac{x}{(x+1)}\ \&$$

$$(x-1)^2Q(x)=(x-1)^2\times-\frac{1}{(x^2-1)}=\frac{(x-1)^2}{(x-1)(x+1)}=-\frac{x-1}{(x+1)}$$

$\Rightarrow x=1, (x-1)P(x) \& (x-1)^2 Q(x)$ *are defined* .

$\Rightarrow x=1$ is a Regular singular point.

Series solution about an ordinary point at $x=x_0$:

A point $x=x_0$ is an ordinary point of the differential equation

$$y'' + p(x) + y' + Q(x)y = 0$$

If y, y', y'' are regular (I. e. analytic and single-valued) there. The general solution near such an ordinary point can be represented by a Taylor series i. e.

$$y(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Example: Find the series solution for $y''+y=0$ at $x=0$ (1)

Solution : comparing the given equation with $y''+p(x)y'+Q(x)y=0$

Here $p(x)=0, Q(x)=1$

Therefore at $x=0, p(x), Q(x)$ are analytic at $x=0$

$\Rightarrow x=0$ is an ordinary point

Let $y=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$ (2)

Be a solution of (1)

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1}\ \&\ y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Eq.(1) become, $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\{(n+2)(n+1)a_{n+2}x^n + a_n\}x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, n = 0,1,2,..........$$

n=0: $a_2 = \frac{1}{(0+2)(0+1)}a_0 = -\frac{1}{(2)(1)}a_0 = -\frac{1}{2!}a_0$;

n=1: $a_3 = \frac{1}{(2+1)(1+1)}a_1 = -\frac{1}{(3)(2)}a_1 = \frac{1}{3!}a_1$;

n=2: $a_4 = -\frac{1}{(2+2)(2+1)}a_2 = -\frac{1}{(4)(3)}a_2 = -\frac{1}{(4)(3)} \times \frac{1}{2!}a_0 = \frac{1}{4!}a_0$;

n=3: $a_5 = -\frac{1}{(3+2)(3+1)}a_3 = -\frac{1}{(5)(4)}a_3 = -\frac{1}{(5)(4)}a_3 = -\frac{1}{(5)(4)} \times \frac{1}{3!}a_1 = \frac{1}{5!}a_1$;

and so on ....

substitute the values in equation Eq. (2)

i.e. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

$$= a_0 + a_1 x + \left(-\frac{1}{2!}a_0\right)x^2 + \left(-\frac{1}{3!}a_1\right)x^3 + \left(\frac{1}{4!}a_0\right)x^4 + \left(\frac{1}{5!}a_1\right)x^5........$$

$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

$$= a_0 \cos x + a_1 \sin x$$

So $y = a_0 \cos x + a_1 \sin x$ is the required solution of equation (1).

Example: Find the series solution at the origin of differential equation

$$(x-1)y'' + 2y' = 0 \qquad (1)$$

Solution: Rewrite the above equation as $\quad y'' + \frac{2}{x-1}y' = 0$

Comparing the given equation with $y'' + P(x)y' + Q(x)y = 0$

Here $\quad P(x) = \frac{2}{x-1}, Q(x) = 0$

Therefore at $x = 0$, $P(x)$, $Q(x)$ are analytic at $x = 0$.

$\Rightarrow x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 a_1 x + a_2 x^2 + a_3 x^3 + ...$

Be a solution of Eq. (1). $\therefore$ $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ & $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

Eq.(1) becomes, $(x-1)\sum_{n=2}^{\infty} n(n+1) a_n x^{n-2} + 2\sum_{n=2}^{\infty} n a_n x^{n-1} = 0$

$$\Rightarrow x\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - \sum_{n=0}^{\infty} \{(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1}\} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - (0+2)(0+1) a_2 - 2(0+1) a_{n+1}\} x^0 = 0$$

$$-\sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - (0+2)(0+1) a_2 - 2(0+1) a_{n+1}\} x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1) n a_{n+1} - \{(n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1}\}\right] x^n \; - \{(2)(1) a_2 - 2(1) a_1\} x^0 = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1)na_{n+1}-(n+2)(n+1)a_{n+2}+2(n+1)a_{n+1}\right]x^n \; -\{2a_2-2a_1\}x^0=0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\{(n+1)n-2(n+1)\}a_{n+1}-(n+2)(n+1)a_{n+2}\right]x^n \; -\{2a_2-2a_1\}x^0=0$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+1}-(n+2)(n+1)a_{n+2}-(n+2)(n+1)a_{n+2}\right]x^n \; -\{2a_2-2a_1\}x^0=0$$

Equating various power of x

i.e, $2a_2-2a_1=0,(n+1)(n+2)a_{n+1}-(n+2)(n+1)a_{n+2}=0$

$$\Rightarrow 2(a_2-a_1)=0,(n+1)(n+2)(a_{n+1}-a_{n+2})=0$$

$$\Rightarrow a_2-a_1=0, a_{n+1}-a_{n+2}=0$$

$$\Rightarrow a_2=a_1, a_{n+2}=a_{n+1} \; n\geq 1$$

Since, $a_{n+2}=a_{n+1}$

$n=1;\; a_3=a_2=a_1(\because a_2=a_1);$

$n=2;\; a_4=a_3=a_1(\because a_3=a_1);$

$n=3;\; a_5=a_4=a_1(\because a_4=a_1);$

$n=4;\; a_6=a_5=a_1(\because a_5=a_1);$

Substitute the values in equation Eq. (2)

i.e

$$\begin{aligned}
y=\sum_{n=0}^{\infty} a_n x^n &= a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ldots\ldots \\
&= a_0+a_1x+a_1x^2+a_1x^3+a_1x^4+a_1x^5+\ldots\ldots\ldots\ldots \\
&=a_0+a_1\left(x+x^2+x^3+x^4+x^5+\ldots\ldots\ldots\right) \\
&=a_0+a_1x\left(1+x+x^3+x^4+\ldots\ldots\ldots\right) \\
&=a_0+a_1\frac{x}{1-x},\left(\because \frac{x}{1-x}=1+x+x^3+x^4+\ldots\ldots\ldots\right)
\end{aligned}$$

$\therefore y=a_0+a_1\frac{x}{1-x}$ is the required solution of equation Eq. (1).

Example: Find the series solution for $\left(1+x^2\right)y''+xy'-y=0\, at\, x=0.(1)$

Solution: Rewrite the above equation as $y''+\frac{x}{\left(1+x^2\right)}y'-\frac{1}{\left(1+x^2\right)}y=0$

Comparing the given equation with $y'' + P(x)y' + Q(x)y = 0$

Here $P(x) = \frac{x}{(1-x^2)}, Q(x) = -\frac{1}{(1-x^2)}$

Therefore at $x = 0$, $P(x), Q(x)$ are analytic at $x = 0$.

$\Rightarrow x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ (2)

Be a solution of (1)

$$\therefore y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \;\&\; y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Eq.(1) becomes, $(1-x^2)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow (n+2)(n-1) a_{0+2} x^0 + (1+2)(1+1) a_{0+2} x^1 + \sum_{n=2}^{\infty} (n+2)(n-1) a_{n+2} x^n$$

$$+\sum_{n=2}^{\infty} n(n-1) a_n x^n + (1)a_1 x^1 + \sum_{n=2}^{\infty} n a_n x^n - \left(a_0 x^0 + a_1 x^1 + \sum_{n=0}^{\infty} a_n x^n \right) = 0$$

$$\Rightarrow 2a_2 x^0 + (3)(2)a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

$$+a_1 x + \sum_{n=2}^{\infty} n a_n x^n - a_0 x^0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow 2a_2 x^0 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n - a_0 x^0 - a_1 x - \sum_{n=2}^{\infty} a_n x^n = 0$$

$$\Rightarrow (2a_2 + a_0) x^0 + (6a_3 + a_1 - a_1) x + \sum_{n=2}^{\infty} \{(n+2)(n+1) a_{n+2} + [n(n-1) + n - 1] a_n\} x^n = 0$$

$$\Rightarrow (2a_2 - a_0) x^0 + (6a_3) x + \sum_{n=2}^{\infty} \{(n+2)(n+1) a_{n+2} + [n(n-1) + n - 1] a_n\} x^n = 0$$

$$\Rightarrow (2a_2 - a_0)x^0 + (6a_3)x + \sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} + [n^2 - n + n - 1]a_n\}x^n = 0$$

$$\Rightarrow (2a_2 - a_0)x^0 + (6a_3)x + \sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} + [n^2 - 1]a_n\}x^n = 0$$

Equating various power of x

i.e. $2a_2 - a_0 = 0, 6a_3 = 0, (n+2)(n+1)a_{n+2} + [n^2 - 1]a_n = 0$

$$\Rightarrow 2a_2 = a_0, a_3 = 0, (n+2)(n+1)a_{n+2} - [n^2 - 1]a_n$$

$$\Rightarrow a_2 = \frac{a_0}{2}, a_3 = 0, a_{n+2} - \frac{[n^2 - 1]a_n}{(n+2)(n+1)}, n \geq 2$$

Now,

$$a_{n+2} = -\frac{[n^2 - 1]a_n}{(n+2)(n+1)} = -\frac{(n-1)(n+1)a_n}{(n+2)(n+1)} = -\frac{(n-1)a_n}{(n+2)}, n = 2, 3, \ldots\ldots\ldots\ldots,$$

This is the recurrence relation.

$$n = 2 : a_4 = \frac{(2-1)a_2}{(2+2)} = -\frac{1}{4}a_2 = -\frac{1}{4}\frac{a_0}{2} = -\frac{a_0}{8}\left(\because a_2 = \frac{a_0}{2}\right);$$

$$n = 3 : a_5 = -\frac{(3-1)a_3}{(3+2)} = -\frac{2}{5}a_3 = 0 (\because a_3 = 0);$$

$$n = 4 : a_6 = -\frac{(4-1)a_4}{(4+2)} = -\frac{3}{6}a_4 = -\frac{1}{2} \times -\frac{a_0}{8} = \frac{a_0}{16}\left(\because a_4 = -\frac{a_0}{8}\right)$$

Substitute the values in equation Eq. (2)

i.e. $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots$

$$= a_0 + a_1 x + \left(\frac{1}{2}a_0\right)x^2 + (0)x^3 + \left(-\frac{a_0}{8}\right)x^4 + (0)x^5 + \left(\frac{a_0}{16}\right)x^6 + \ldots\ldots\ldots$$

$$= a_0\left(1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} + \ldots\ldots\right) + a_1(x)$$

$\therefore y = a_0\left(1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} + \ldots\ldots\right) + a_1(x)$ is the required solution of Eq. (1)

EXERCISE

1. Find the power series solution of the equation $y'' + y' + y = 0$
2. Find the general solution of $(1 + x^2)y'' + 2xy' - 2y = 0$ in term of power series in x.

Ans:

$$1.\, y = a_0\{1 - \frac{x^2}{2} + \frac{x^4}{2.4} - ....\} + a_1\{x - \frac{x^3}{3} + \frac{x^5}{3.5} - ....\}$$

$$2.\, y = a_0\{1 + x^2 - \frac{x^4}{3} - ....\} + a_1 x$$

Series solution about regular singular point at $x = x_0$:

A point $x = x_0$ is a regular point of the differential equation

$$y^{''} + P(x)y^{'} + Q(x)y = 0$$

If $y, y^{'}, y^{''}$ are not all regular there, but $(x - x_0)y^{'}, (x - x_0)^2 y$ are all regular at x_0 .This essentially implies that y(x) must have a fixed order divergence (or pole) at x_0 .The general solution near such a regular singular point can be represented by a **Frobenius series**

$$y(x) = x^m \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

With $a_0 \neq 0$ and note that m is not necessarily an integer.

Example: Find the series solution of $4xy^{''} + 2y^{'} + y = 0$ by Frobenius method

Solution: Consider the ode $4xy^{''} + 2y^{'} + y = 0$ (1)

Now, we can rewrite the above Eq. (1) as $y^{''} + \frac{1}{2x}y^{'} + \frac{1}{4x}y = 0$

Comparing this with $y^{''} + P(x)y^{'} + Q(x)y = 0,$

Here $p(x) = \frac{1}{2x},\ Q(x) = \frac{1}{4x}.$

Therefore, at x = 0 is not an ordinary point i.e., it is singular point.

Now, $y^{''} + xP(x) = \frac{1}{2}, x^2 Q(x) = \frac{x}{4} \Rightarrow xP(x), x^2 Q(x)$ are analytic at x=0

$\Rightarrow x = 0$ is a regular singular point.

We use **Frobenius method** for the solution of equation Eq. (1)

Let $y = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n}$ (2)

Be a solution of Eq. (1).

$$\therefore y'' = \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} \;\&\; \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2}$$

Eq. (1) becomes,

$$4x\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{m+n-2} + 2\sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}4(m+n)(m+n-1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}2(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\{4(m+n)(m+n-1)+2(m+n)\}a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)\{2(m+n-1)+1\}a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)\{2m+2n-2+1\}a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)(2m+2n-2+1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}2(m+n)(2m+2n-1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)(2m+2(n+1)-1)a_{n+1} x^{m+n+1-1} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)(2m+2(n+1)-1)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)(2m+2n+2-1)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty}2(m+n+1)(2m+2n+1)a_{n+1} x^{m+n} + \sum_{n=0}^{\infty}a_n x^{m+n} = 0$$

$$\Rightarrow 2\left(m-n+1\right)(2m+2(-1)+1)a_0x^{m-1}+\sum_{n=-1}^{\infty}2\left(m+n+1\right)(2m+2n+1)a_{n+1}x^{m+n}+\sum_{n=0}^{\infty}a_nx^{m+n}=0$$

$$\Rightarrow 2(m)\left(2m-2+1\right)a_0x^{m-1}+\sum_{n=-1}^{\infty}\{2\left(m+n+1\right)(2m+2n+1)a_{n+1}+a_n\}x^{m+n}=0$$

$$\Rightarrow 2(m)\left(2m-1\right)a_0x^{m-1}+\sum_{n=-1}^{\infty}\{2\left(m+n+1\right)(2m+2n+1)a_{n+1}+a_n\}x^{m+n}=0$$

Equating to zero the coefficient of lowest power and highest power i.e.

$$x^{m-1}:2(m)(2m-1)a_0=0 \tag{3}$$

Which called an **Indicial equation** &

$$x^{m+n}:2(m+n+1)(2m+2n+1)a_{n+1}+a_n=0 \tag{4}$$

Which is called **Recurrence relation.**

Solving eq. (3) i.e.

$$2(m)\left(2m-1\right)a_0=0\Rightarrow m\left(2m-1\right)=0\left(\because a_0\neq 0\right)$$

$$\Rightarrow m=0, m=\frac{1}{2}.$$

Solving Eq. (4) i.e.

$$2\left(m+n+1\right)\{2m+2n+1\}a_{n+1}-\{2m+2n+1\}a_n=0$$

$$\Rightarrow a_{n+1}=-\frac{1}{2\left(m+n+1\right)(2m+2n+1)}a_n, n\geq 0$$

When m=0, then the equation Eq. (5) reduces to

$$\Rightarrow a_{n+1}=-\frac{1}{2\left(n+1\right)(2n+1)}a_n, n\geq 0$$

Putting the values of n=0,1,2,…………..

$$n=0: a_1=-\frac{1}{2(1)(1)}a_0=-\frac{1}{2}a_0=-\frac{1}{2!}a_0,$$

$$n=1: a_2=-\frac{1}{2(1+1)(2\times1+1)}a_1=-\frac{1}{(4)(3)}a_1=-\frac{1}{(4)(3)}\times-\frac{1}{2!}a_0=\frac{1}{4!}a_0$$

$$n=2: a_3=-\frac{1}{2(2+1)(2\times2+1)}a_2=-\frac{1}{2(3)(5)}a_2=-\frac{1}{(6)(5)}\times\frac{1}{4!}a_0=-\frac{1}{6!}a_0$$

And so on.

Substitute these values in Eq. (2)

$$y=x^m\sum_{n=0}^{\infty}a_nx^n=x^m\left(a_0+a_1x+a_2x^2+a_3x^3+........\right)$$

$$y=x^0\left\{a_0+\left(-\frac{1}{2!}a_0\right)x+\left(\frac{1}{4!}a_0\right)x^2+\left(-\frac{1}{6!}a_0\right)x^3+......\right\}$$

$$y=a_0\left\{1-\frac{1}{2!}x+\frac{1}{4!}x^2-\frac{1}{6!}x^3+......\right\}$$

One solution of the given equation is

$$u=1-\frac{1}{2!}x+\frac{1}{4!}x^2-\frac{1}{6!}x^3+......\left(taking\ a_0=1\right)\quad(6)$$

When $m=\frac{1}{2}$,then the equation Eq. (5) reduces to

$$\Rightarrow a_{n+1}=-\frac{1}{2\left(\frac{1}{2}+n+1\right)\left(2\times\frac{1}{2}+2n+1\right)}a_n, n\geq 0$$

$$=-\frac{1}{(1+2n+2)(1+2n+1)}a_n$$

$$=-\frac{1}{(2n+3)(2n+2)}a_n$$

Putting the values of n=0,1,2,………..

$$n=0: a_1=\frac{a_0}{1+2(0)+2}=\frac{a_0}{1+2}=\frac{a_0}{3}$$

$$y=au+bv$$

$$n=1: a_2=\frac{a_1}{1+2(1)+2}=\frac{a_1}{1+2+2}=\frac{a_1}{5}=\frac{1}{5}\frac{a_0}{3}=\frac{a_0}{15}\left(\because a_1=\frac{a_0}{3}\right)$$

$$n=2: a_3=\frac{a_2}{1+2(2)+2}=\frac{a_2}{1+4+2}=\frac{a_2}{7}=\frac{1}{7}\times\frac{a_0}{15}=\frac{a_0}{105}\left(\because a_2=\frac{a_0}{15}\right)$$

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$$v=x^{\frac{1}{2}}\left[1+\left(\frac{1}{3}\right)x+\left(\frac{1}{15}\right)x^2+\left(\frac{1}{105}\right)x^3+....\right](taking\ a_0=1)$$

$$n=1: a_2=-\frac{1}{(2+1)(2+2)}a_1=-\frac{1}{(5)(4)}a_1=-\frac{1}{(5)(4)}\times-\frac{1}{3!}a_0=\frac{1}{5!}a_0$$

$$n=2: a_3=-\frac{1}{(2\times2+3)(2\times2+2)}a_2=-\frac{1}{(7)(6)}a_2=-\frac{1}{(7)(6)}\times\frac{1}{5!}a_0=-\frac{1}{7!}a_0$$ and so on.

Substitute these values in Eq. (2)

$$y=x^m\sum_{n=0}^{\infty}a_nx^n=x^m\left(a_0+a_1x+a_2x^2+a_3x^3+........\right)$$

$$y=x^{\frac{1}{2}}\left\{a_0+\left(-\frac{1}{3!}a_0\right)x+\left(\frac{1}{5!}a_0\right)x^2+\left(-\frac{1}{7!}a_0\right)x^3+......\right\}$$

$$y=a_0\left(x^{\frac{1}{2}}\left\{1-\left(-\frac{1}{3!}\right)x+\left(\frac{1}{5!}\right)x^2+\left(-\frac{1}{7!}\right)x^3+......\right\}\right)$$

Another solution of the given equation is

$$v=x^{\frac{1}{2}}\left\{1-\left(\frac{1}{3!}\right)x+\left(\frac{1}{5!}\right)x^2+\left(\frac{1}{7!}\right)x^3+......\right\}$$ (taking $a_0=1$)

Therefore, the complete solution of the differential equation is

$y=au+bv$, where u and v are given in equation Eq. (6) and (7) respectively.

Example: solve the differential equation

$4xy''+2(1-x)y'-y=0$ by Frobenius method of power series solution.

Solution: consider the differential equation $4xy''+2(1-x)y'-y=0$ (1)

Now, we can rewrite the above Eq. (1) as

$$y''+\frac{(1-x)}{2x}y'-\frac{1}{4x}y=0$$
$$P(x)=\frac{(1-x)}{2x}, Q(x)=-\frac{1}{4x}$$

Comparing this with $y''+P(x)y'+Q(x)y=0$

Here $P(x)=\frac{(1-x)}{2x}, Q(x)=-\frac{1}{4x}$.

Therefore at x=0 , Q(x) is not analytic at $x=0$.

$\Rightarrow x=0$ is not ordinary point i.e. it is singular point.

Now, $xP(x)=\frac{(1-x)}{2}, x^2Q(x)=-\frac{x}{4} \Rightarrow xP(x), x^2Q(x)$ are analytic at $x=0$

$\Rightarrow x=0$ is a regular singular point.

We use Frobenius method for the solution of equation Eq. (1)

Let $y=x^m\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{m+n}$ (2)

$$\therefore y''=\sum_{n=0}^{\infty}(m+n)a_nx^{m+n-1} \,\&\, \sum_{n=0}^{\infty}(m+n)(m+n-1)a_nx^{m+n-2}$$

$$m=0 \,\&\, m=\frac{1}{2}$$

Solving Eq.(4) i.e.

$$2\left(m+n+1\right)\{2m+2n+1\}a_{n+1}-\{2m+2n+1\}a_n=0$$

$$\Rightarrow\left(2m+2n+1\right)[2\{m+n+1\}a_{n+1}-a_n]=0$$

$$\Rightarrow 2\{m+n+1\}a_{n+1}-a_n=0$$

$$\Rightarrow a_{n+1}=\frac{a_n}{2\{m+n+1\}}, n\geq 0 \tag{5}$$

When m = 0, then the equation Eq. (5) reduces to

$$\Rightarrow a_{n+1}=\frac{a_n}{2\{0+n+1\}}, n\geq 0$$

$$\Rightarrow a_{n+1}=\frac{a_n}{2(n+1)}, n\geq 0$$

Putting the value of n=0,1,2,………….

$$n=0: a_1=\frac{1}{2\left(0+1\right)}a_0=\frac{1}{2}a_0$$

$$n=1: a_2=\frac{1}{2\left(1+1\right)}a_1=\frac{1}{2\left(2\right)}a_1=\frac{1}{4}a_1=\frac{1}{4}\times\frac{1}{2}a_0=\frac{1}{8}a_0\left(\because a_1=\frac{1}{2}a_0\right)$$

$$n=2: a_3=\frac{1}{2\left(2+1\right)}a_2=\frac{1}{2\left(3\right)}a_2=\frac{1}{6}a_2=\frac{1}{6}\times\frac{1}{8}a_0=\frac{1}{48}a_0\left(\because a_2=\frac{1}{8}a_0\right)$$

And so on.

Substitute these values in Eq. (2)

$$y=x^m\sum_{n=0}^{\infty}a_nx^n=x^m\left(a_0+a_1x+a_2x^2+a_3x^3+........\right)$$

$$y=x^0\left\{a_0+\left(\frac{a_0}{2}\right)x+\left(\frac{a_0}{8}\right)x^2+\left(\frac{a_0}{48}\right)x^3+......\right\}$$

$$y = a_0\left\{1+\left(\frac{1}{2}\right)x+\left(\frac{1}{8}\right)x^2+\left(\frac{1}{48}\right)x^3+......\right\}$$

One solution of the given equation is

$$u = \left\{1+\left(\frac{1}{2}\right)x+\left(\frac{1}{8}\right)x^2+\left(\frac{1}{48}\right)x^3+......\right\}(taking\, a_0=1) \tag{6}$$

When $m=\frac{1}{2}$,then the equation Eq. (5) reduces to

$$a_{n+1}=\frac{a_n}{2\left\{\frac{1}{2}+n+1\right\}}=\frac{a_n}{2\left\{\frac{1+2n+2}{2}\right\}}=\frac{a_n}{1+2n+2}, n\geq 0$$

Putting the values of n=0,1, 2,…

$$n=0: a_1=\frac{a_0}{1+2(0)+2}=\frac{a_0}{1+2}=\frac{a_0}{3}$$

$$n=1: a_2=\frac{a_1}{1+2(1)+2}=\frac{a_1}{1+2+2}=\frac{a_1}{5}=\frac{1}{5}\frac{a_0}{3}=\frac{a_0}{15}\left(\because a_1=\frac{a_0}{3}\right)$$

$$n=2: a_3=\frac{a_2}{1+2(2)+2}=\frac{a_2}{1+4+2}=\frac{a_2}{7}=\frac{1}{7}\times\frac{a_0}{15}=\frac{a_0}{105}\left(\because a_2=\frac{a_0}{15}\right)$$

And so on.

Substitute these values in Eq. (2)

$$y=x^m\sum_{n=0}^{\infty}a_nx^n=x^m\left(a_0+a_1x+a_2x^2+a_3x^3+........\right)$$

$$y=x^{\frac{1}{2}}\left\{a_0+\left(\frac{a_0}{3}\right)x+\left(\frac{a_0}{15}\right)x^2+\left(\frac{a_0}{105}\right)x^3+......\right\}$$

$$y=a_0\left(x^{\frac{1}{2}}\left\{1+\left(\frac{1}{3}\right)x+\left(\frac{1}{15}\right)x^2+\left(\frac{1}{105}\right)x^3+......\right\}\right)$$

Another solution of the given equation is

$$v = x^{\frac{1}{2}}\left[1+\left(\frac{1}{3}\right)x+\left(\frac{1}{15}\right)x^2+\left(\frac{1}{105}\right)x^3+....\right]\left(taking\ a_0=1\right) \tag{7}$$

Therefore, the complete solution of the differential Eq. (6) and (7)

$y = au + bv$, where u and v are given in equations Eq. (6) and (7) respectively.

Example: Find the series solution of $2x^2y'' - xy' + (x-5)y = 0$

ANS: $u = x^{-1}\left\{1+\left(\frac{1}{5}\right)x+\left(\frac{1}{30}\right)x^2+......\right\}(taking\ a_0=1)$

$$v = x^{\frac{5}{2}}\left\{1-\left(\frac{1}{9}\right)x+\left(\frac{1}{198}\right)x^2+......\right\}\left(taking\ a_0=1\right)$$

EXERCISE

Find the power series solution of the following differential equation by Frobenius method

1. $2xy''+(3-x)y'-y=0$
2. $2x^2y''+xy'-(x+1)y=0$

ANS: $$\left(\begin{array}{l} 1.u=1+\dfrac{x}{1.3}+\dfrac{x}{1.3.5}+........., \quad v=x^{-\frac{1}{2}}\left\{1+x+\dfrac{x^2}{2^2 2!}+......\right\} \\ 2.u=x\left\{1+\dfrac{x}{5}+\dfrac{x^2}{70}+.........,\right\}, \quad v=x^{-\frac{1}{2}}\left\{1-x-\dfrac{x^2}{2}+......\right\} \end{array}\right)$$